Stat 100a: Introduction to Probability.

Outline for the day

1. Review list.
2. Example problems.
3. Tournaments.

Submit HW3 to statgrader!

http://www.stat.ucla.edu/~frederic/100A/sum21 .

Wed is the final exam, 10am to 11:20am.

You must zoom in with your camera on, to this same zoom, for the final!

Again any notes and books are fine.

The exam will be called exam3.pdf on the course website, and you will email me your answers by 11:20am to frederic@stat.ucla.edu .

Review list.

1) Basic principles of counting.
2) Axioms of probability, and addition rule.
3) Permutations & combinations.
4) Conditional probability.
5) Independence.
6) Multiplication rules. $P(AB) = P(A)\ P(B|A)$ [= $P(A)P(B)$ if ind.]
7) Odds ratios.
8) Random variables (RVs).
9) Discrete RVs, and probability mass function (pmf).
10) Expected value.
11) Pot odds calculations.
12) Luck and skill.
13) Variance and SD.
14) Bernoulli RV. [0-1. $\mu = p$, $\sigma = \sqrt{(pq)}$.]
15) Binomial RV. [# of successes, out of n tries. $\mu = np$, $\sigma = \sqrt{(npq)}$.]
16) Geometric RV. [# of tries til 1st success. $\mu = 1/p$, $\sigma = (\sqrt{q}) / p$.]
17) Negative binomial RV. [# of tries til rth success. $\mu = r/p$, $\sigma = (\sqrt{rq}) / p$.]
18) Poisson RV [# of successes in some time interval. [$\mu = \lambda$, $\sigma = \sqrt{\lambda}$.]
19) E(X+Y), V(X+Y) (ch. 7.1).
20) Bayes's rule (ch. 3.4).
20) Continuous RVs, Uniform, Normal, Exponential and Pareto.
21) Probability density function (pdf). Recall $F'(c) = f(c)$, where $F(c)$ = cdf.
22) Moment generating functions
23) Markov and Chebyshev inequalities
24) Law of Large Numbers (LLN) and Fundamental Theorem of Poker.
25) Central Limit Theorem (CLT)
26) Conditional expectation.
27) Confidence intervals for the sample mean and sample size calculations.
28) Random Walks, Reflection Principle, Ballot Theorem, avoiding zero
29) Chip proportions, doubling up, and induction.
30) Bivariate normal distribution and the conditional distribution of Y given X.
31) Covariance and correlation.

Basically, we've done all of ch. 1-7 except 6.7 and the optimal strategy stuff in 6.3.

What integrals do you need to know?

You need to know $\int e^{ax}\,dx$, $\int ax^k\,dx$ for any k, and $\int \log(x)\,dx$,

and basic stuff like $\int [af(x) + g(x)]\,dx = a\int f(x)dx + \int g(x)dx$,

and you need to understand that $\int\int f(x,y)\,dy\,dx = \int [\int f(x,y)\,dy]\,dx$.

Suppose you have 10 players at the table. What is the expected number of players who have 2 face cards? A face card is a J, Q, K.

P(2 face cards) = C(12,2)/C(52,2) = 4.98%.

Let X1 = 1 if player 1 has 2 face cards, and X1 = 0 otherwise.

X2 = 1 if player 2 has 2 face cards, and X2 = 0 otherwise. etc.

$X = \sum X_i$ = total number of players with 2 face cards.

$E(X) = \sum E(X_i) = 10 \times 4.98\% = 0.498$.

Let $X = N(0, 0.8^2)$ and $\varepsilon = N(0, 0.1^2)$ and ε is independent of X. Let $Y = 7 + 0.2\ X + \varepsilon$.

Find E(X), E(Y), E(Y|X), var(X), var(Y), cov(X,Y), and ρ = cor(X,Y).

$E(X) = 0$.

$E(Y) = E(7 + 0.2X + \varepsilon) = 7 + 0.2\ E(X) + E(\varepsilon) = 7$.

$E(Y|X) = E(7 + 0.2X + \varepsilon \mid X) = 7 + 0.2X + E(\varepsilon \mid X) = 7 + 0.2X$ since ε and X are ind.

$\text{var}(X) = 0.64$.

$\text{var}(Y) = \text{var}(7 + 0.2\ X + \varepsilon) = \text{var}(0.2X + \varepsilon) = 0.2^2\ \text{var}(X) + \text{var}(\varepsilon) + 2*0.2\ \text{cov}(X,\varepsilon)$

$= 0.2^2(.64) + 0.1^2 + 0 = 0.0356$.

$\text{cov}(X,Y) = \text{cov}(X, 7 + 0.2X + \varepsilon) = 0.2\ \text{var}(X) + \text{cov}(X, \varepsilon) = 0.2(0.64) + 0 = 0.128$.

$\rho = \text{cov}(X,Y)/(\text{sd}(X)\ \text{sd}(Y)) = 0.128 / (0.8 \times \sqrt{.0356}) = 0.848$.

Suppose (X,Y) are bivariate normal with E(X) = 10, var(X) = 9, E(Y) = 30, var(Y) = 4, ρ = 0.3,

What is the distribution of Y given X = 7?

Given X = 7, Y is normal. Write $Y = \beta_1 + \beta_2 X + \varepsilon$ where ε is normal with mean 0, ind. of X.

Recall $\beta_2 = \rho\, \sigma_y/\sigma_x = 0.3 \times 2/3 = 0.2$.

So $Y = \beta_1 + 0.2\, X + \varepsilon$.

To get β_1, note $30 = E(Y) = \beta_1 + 0.2\, E(X) + E(\varepsilon) = \beta_1 + 0.2\, (10) + 0$. So $30 = \beta_1 + 2$. $\beta_1 = 28$.

So $Y = 28 + 0.2\, X + \varepsilon$, where ε is normal with mean 0 and ind. of X.

What is var(ε)?

$4 = \text{var}(Y) = \text{var}(28 + 0.2\, X + \varepsilon) = 0.2^2\, \text{var}(X) + \text{var}(\varepsilon) + 2(0.2)\, \text{cov}(X,\varepsilon)$

$= 0.2^2\, (9) + \text{var}(\varepsilon) + 0$. So $\text{var}(\varepsilon) = 4 - 0.2^2(9) = 3.64$ and $\text{sd}(\varepsilon) = \sqrt{3.64} = 1.91$.

So $Y = 28 + 0.2\, X + \varepsilon$, where ε is $N(0, 1.91^2)$ and ind. of X.

Given X = 7, $Y = 28 + 0.2(7) + \varepsilon = 29.4 + \varepsilon$, so $Y|(X=7) \sim N(29.4, 1.91^2)$.

Bivariate and marginal density example.

Suppose the joint density of X and Y is $f(x,y) = a(xy + x + y)$, for X and Y in (0,2) x (0,2). What is a? What is the marginal density of Y? What is the density of X conditional on Y? What is E(X|Y)? Are X and Y independent?

$\iint a(xy + x + y)\, dy\, dx = 1 = a\int [xy^2/2 + xy + y^2/2\ ]_{y=0}^{2}\, dx = a\int [2x + 2x + 2 - 0 - 0 - 0]dx$
$= a(x^2 + x^2 + 2x)\ ]_{x=0}^{2} = a(4 + 4 + 4 - 0 - 0 - 0) = 12a$, so $a = 1/12$.
The marginal density of Y is $f(y) = \int_0^2 a(xy+x+y)\, dx$
$= ay \int_0^2 x dx + a\int_0^2 x dx + ay \int_0^2 dx$
$= y/12\ (x^2/2)\ ]_{x=0}^{2} + 1/12\ (x^2/2)]_{x=0}^{2} + y/12\ x]_{x=0}^{2}$
$= 2y/12 + 2/12 + 2y/12$
$= y/3 + 1/6$.
Check that this is a density. $\int_0^2 (y/3 + 1/6)\, dy = (y^2/6 + y/6)\ ]_{y=0}^{2} = 4/6 + 2/6 - 0 - 0 = 1$.

Conditional on Y, the density of X is $f(x|y) = f(x,y)/f(y) = (xy+x+y) / [12(y/3+1/6)]$
$= (xy+x+y)/(4y+2)$.
$E(X|Y) = \int_0^2 x(xy+x+y)/(4y+2)\, dx = (x^3y/3 + x^3/3 + x^2y/2)/(4y+2)]_{x=0}^{2}$
$= (8y/3+8/3+2y-0-0-0)/(4y+2) = (14y/3 + 8/3)/(4y+2)$.
$f(y) = y/3 + 1/6$ and similarly $f(x) = x/3 + 1/6$,
so $f(x)f(y) = xy/9 + x/18 + y/18 + 1/36 \neq f(x,y)$. So, X and Y are not independent.

Let X = 1 if you are dealt pocket aces and 0 otherwise. Let Y = 1 if you are dealt two black cards and 0 otherwise. What is cov(3X, 7Y)?

cov(3X, 7Y) = 21cov(X,Y).
cov(X,Y) = E(XY) – E(X)E(Y).
E(X) = 1 P(pocket aces) + 0 P(not pocket aces) = C(4,2)/C(52,2) = 0.452%.
E(Y) = 1 P(2 black cards) + 0 P(not 2 black cards) = C(26,2)/C(52,2) = 24.5%.
Here XY = 1 if X and Y are both 1, and XY = 0 otherwise.
So E(XY) = 1 P(X and Y = 1) + 0 P(X or Y does not equal 1)
= P(2 black aces) + 0
= 1 / C(52,2) = 0.0754%.
cov(X,Y) = .000754 - .00452(.245) = -.0003534.
cov(3X, 7Y) = 21 (-.0003534) = -.00742.

Suppose X = the number of hands until you get dealt at least one black card. After this, you play 100 more hands and Y = the number of hands where you get dealt pocket aces out of these next 100 hands.

Let Z = 4X + 7Y. What is the SD of X? What is SD(Y)? What is E(Z)? What is SD(Z)?

X is geometric(p), where p = 1 – P(both red) = 1 – C(26,2)/C(52,2) ~ 75.5%. SD(X) = √q/p = 0.656.

Y is binomial(n,p), n = 100 and p = C(4,2)/C(52,2) ~ 0.452%. SD(Y) = √(npq) = 0.671.

E(Z) = 4E(X) + 7E(Y) = 4(1/.755) + 7(100)(.00452) = 8.46.

X and Y are independent so Var(Z) = Var(4X) + Var(7Y) = 16Var(X) + 49Var(Y) = 16(.656^2) + 49(.671^2) = 28.9. So SD(Z) = √28.9 = 5.38.

CLT Example

Suppose X1, X2, ..., X100 are 100 iid draws from a population with mean μ=70 and sd σ=10. What is the approximate distribution of sample mean $\bar{x}$?

By the CLT, the sample mean is approximately normal with mean μ and sd $\sigma/\sqrt{n}$, i.e. $\sim N(70, 1^2)$.

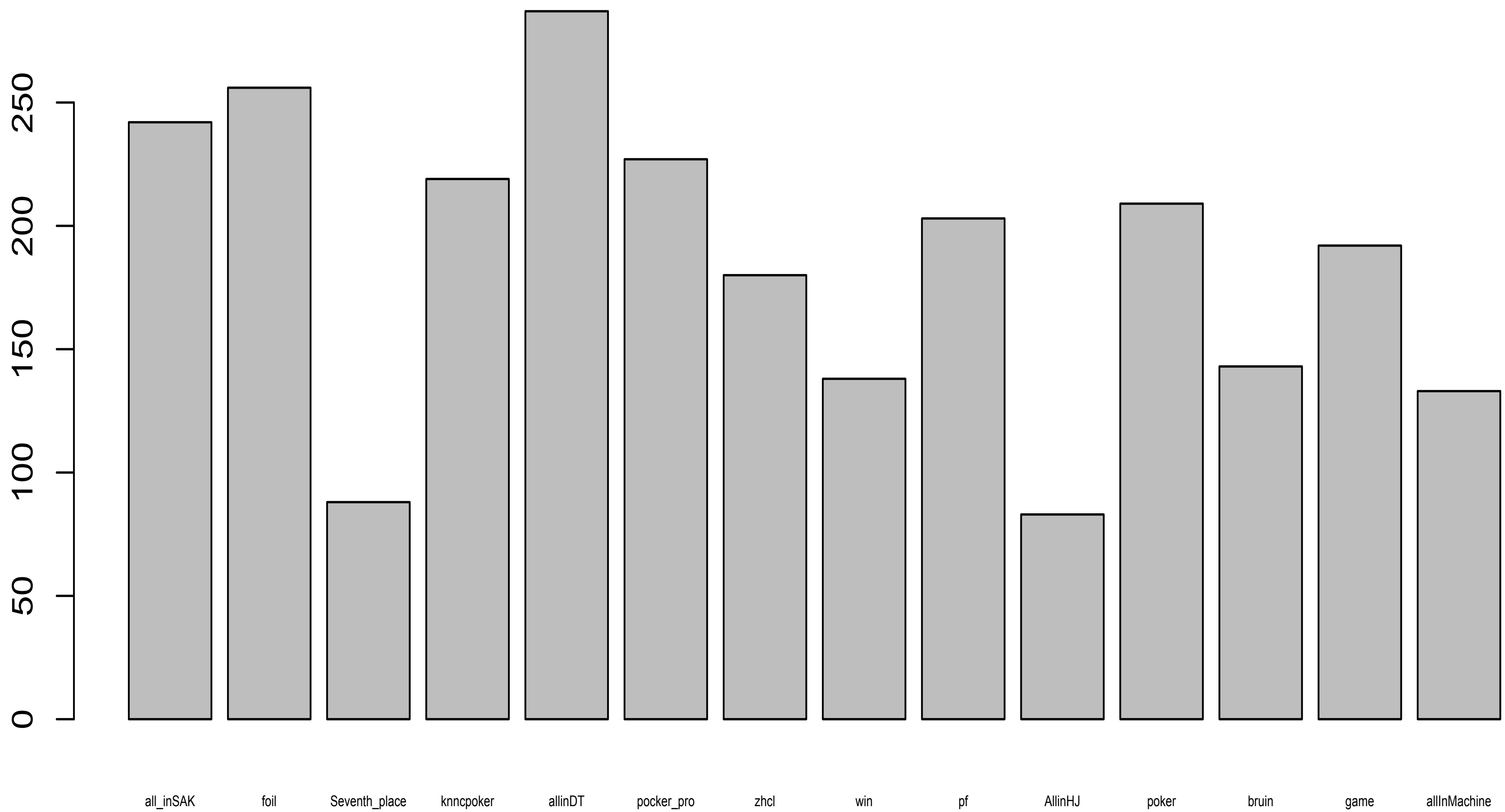
Now suppose Y1, Y2, ..., Y100 are iid draws, independent of X1, X2, ..., X100, with mean μ=80 and sd σ=25. What is the approximate distribution of $\bar{x} - \bar{y} = Z$?

Now the sample mean of the first sample is approximately $N(70, 1^2)$ and similarly the negative sample mean of the 2nd sample is approximately $N(-80, 2.5^2)$, and the two are independent, so their sum Z is approximately normal.

Its mean is 70-80 = -10,

and $var(Z) = 1^2+2.5^2 = 7.25$, so $Z \sim N(-10, 2.69^2)$, because $2.69^2 = 7.25$.

Remember, if X and Y are ind., then var(X+Y) = var(X) + var(Y).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