

1. Miscellaneous
2. Conditional intensities for point processes.
3. Poisson processes.

1) Miscellaneous

- a) Projects. Interpret your coefficients!
- b) Kernel intensity estimation from last lecture.

Kernel density estimation.

$$\hat{f}(x) = 1/(n h) \sum_{i=1}^n K[(x - x_i)/h].$$

Kernel intensity estimates.

$$\hat{\lambda}(x) = 1/h \sum_{i=1}^n K[(x - x_i)/h].$$

```
x1 = sort(c(runif(1000)*30,rnorm(300)*2+10))
```

```
hist(x1,nclass=50,prob=F,main="kernel intensity estimate")
```

```
z1 = density(x1,bw=bw.nrd0(x1))
```

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lines(z1$x,z1$y*1300/z1$bw,lty=1,col=4)
```

2) Conditional intensities for point processes.

$\lambda(t) = \lim_{\Delta t \rightarrow 0} E[N(t, t + \Delta t) | H_t] / \Delta t =$ rate at which we expect points to accumulate around time t , given info on what points have occurred prior to t .

If, for disjoint sets B_1, B_2, \dots, B_j , the number of points in these sets are independent of each other, then $E\lambda = \lambda$. If in addition N is simple, then N is called a Poisson process.

3) Poisson processes.

Suppose N is simple, and $N(B_1), N(B_2), \dots, N(B_j)$, are independent random variables, provided B_1, B_2, \dots, B_j are disjoint sets. Then N is a Poisson process. The name

comes from the fact that it follows that for any set B , $N(B)$ has a Poisson distribution with some mean $\mu(B)$. That is:

$$P[N(B) = k] = \mu(B)^k e^{-\mu(B)} / k!, \quad \text{for } k = 0, 1, 2, \dots$$

For a Poisson process, $E\lambda = \lambda$, so we can call them both the "intensity".

How does the intensity λ relate to the mean $\mu(B)$?

When λ exists and is finite, the mean $\mu(B)$ is simply the integral of the overall intensity:

$$\mu(B) = \int_B \lambda(t) dt.$$

$\lambda(t)$ is the expected number of points per unit time and $\mu(B)$ is the expected number of points in the time interval B .

If N is a Poisson process, does that mean that $\lambda(t)$ is constant for all t ?

No. It just means that $\lambda(t)$ doesn't depend on what other points have occurred. The rate could be high at some times and low at other times, regardless of what other points have happened. If the rate is constant, then the process is called *stationary* or *homogeneous*.

What does it mean for a point process to be a stationary Poisson process?

It means that it is simple and that given that there are exactly k points in B , those k points are distributed *uniformly* within B . In other words, a stationary Poisson process has its points uniformly scattered throughout the space S .

In fact, a point process obeys these conditions (simplicity and uniformity) if and only if it is a stationary Poisson process.

What's an example of an *inhomogeneous* Poisson process?

To specify a Poisson process, all you need to do is specify the rate $\lambda(t)$. For instance, suppose the space S is a portion of the real time line, $[0,10]$. You could have $\lambda(t) = t+2.5$, or $t^2 + 4t + 2$. In both of these cases we'd expect few points near the origin, and lots more points as time increases.

What about $t^2 + 4t - 2$?

The rate can't be negative.

If instead $\lambda(t) = 2.5$, then it's a homogeneous Poisson process, and we expect 2.5 points per unit time.

e) What's an example of a non-Poisson point process?

It may be self-exciting, or self-correcting. That is, the occurrence of points in some time interval B_1 may influence the number of points that will occur in a nearby interval B_2 . Remember, in order for the point process to be Poisson, $N(B_1)$ must be independent of

$N(B_2)$, for disjoint sets B_1 and B_2 . Self-exciting = clustered. Self-correcting = inhibitory.

What's an example of a self-exciting point process?

Poisson cluster process: start with a Poisson process, and for each point in it, consider that point a "parent". For each parent, generate a random number of "children", so that the parents' numbers of children are iid from some probability distribution P . Let the spatial locations of the children be dispersed around the parent, independently of each other, according to some spatial density f .

Then look at the final map of all the children.

This is a self-exciting process: if you know there are points in B_1 , then it's more likely that there will be points in a nearby region B_2 .

What's a simple example of a self-correcting process?

Strauss hard-core process: start with a Poisson process, and pick some number δ . Find all pairs of points that are within δ of one another, and delete all such points. The remaining points must be well-dispersed. If you know there's a point in B_1 , then that tells you it is less likely that there's a point in a nearby region B_2 .