Statistics 222, Spatial Statistics.

Outline for the day:

1. Simple, orderly.
2. Conditional intensity.
3. Poisson process.

1. Simple and orderly point processes.

Last time we noted that the accepted definition of a point process is a Z+ valued random measure.

We will sometimes use the notation τ_i to represent point i = (t_i, x_i, y_i).

A point process is *stationary* (or *homogeneous)* if, for any k and any collection of measurable sets, B_1, B_2, ..., B_k, the joint distribution of the collection $\{N(B_1+\Delta), N(B_2+\Delta), ..., N(B_k+\Delta)\}$ doesn't depend on Δ.

A point process is called *simple* if all the points are distinct, i.e.

P(there are indices i and j where $\tau_i = \tau_j$) = 0.

A point process is *orderly* if for any time t and any spatial interval B,

$\lim_{\Delta t \to 0} P(N([t,t+\Delta t) \times B) > 1) / (\Delta t\, |B|) = 0.$

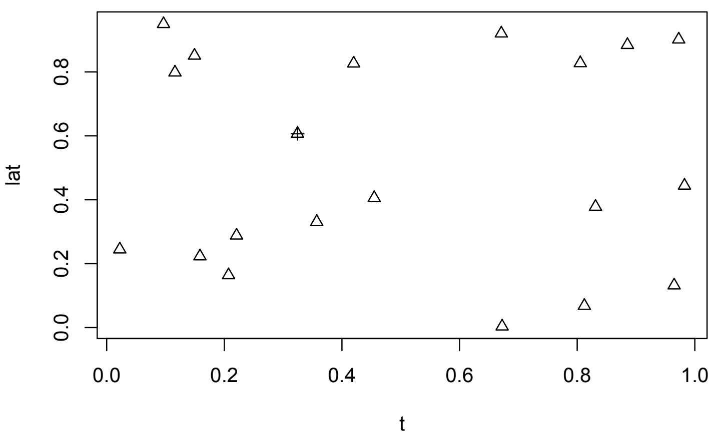
If *N* is simple and stationary, then it is orderly (Daley and Vere-Jones, 2003, 3.3.V and 3.3.VI).

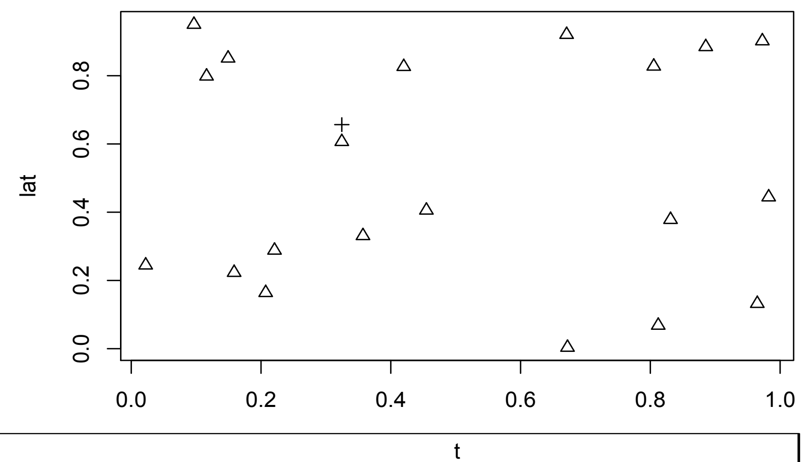
1. Simple and orderly point processes, continued.

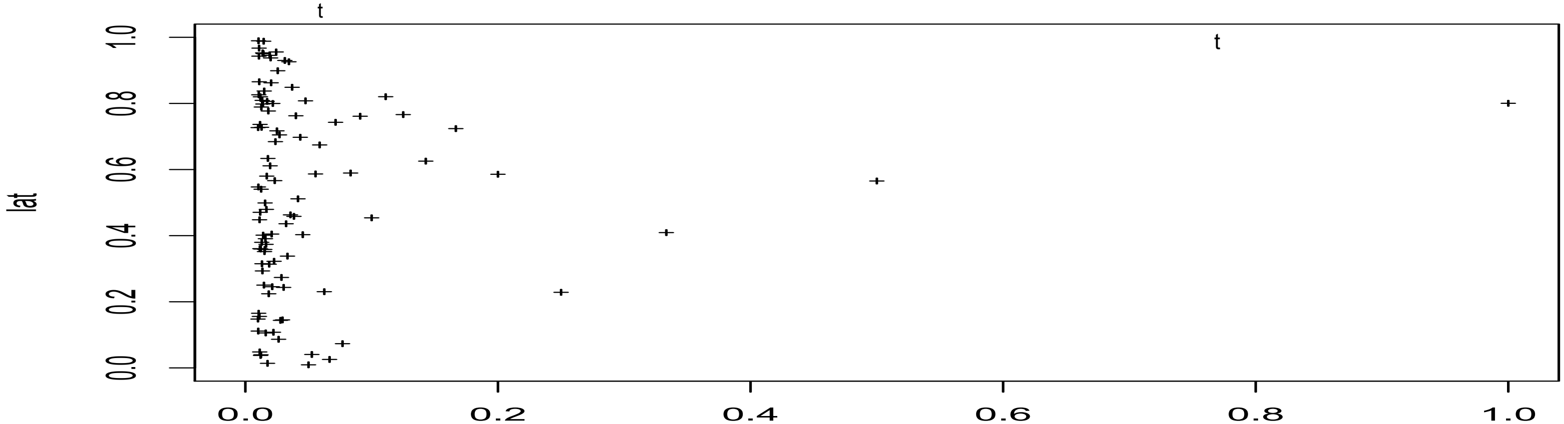
The times $\{t_1, t_2, \ldots, t_n\}$ are sometimes said to form the *ground* process.

N has *simple ground process* if all the *times* are distinct, with prob. 1.

Examples of a non-simple process, *N* with non-simple ground process, and a nonorderly process with points at $\{(1/i, U(0,1), i=1,2,\ldots\}$.







2. Conditional intensity, λ.

Fix any space-time location (t,x,y). $\lambda(t,x,y)$ is the limiting expected rate of accumulation of points around (t,x,y), given all points prior to t.

$\lambda(t,x,y) = \lim_{\Delta t,\, \delta \to 0} E(N([t,t+\Delta t) \times B(x,y,\delta)) \mid H_t) / (\Delta t\, \pi\delta^2)$, where

B(x,y,δ) is a circle of radius δ around (x,y), and H_t is the history of the process up to but not including time t.

If N is orderly, then

$$\lambda(t,x,y) = \lim_{\Delta t,\, \delta \to 0} P(N([t,t+\Delta t) \times B(x,y,\delta)) > 0 \mid H_t) / (\Delta t\, \pi\delta^2).$$

Note that λ is random. It can depend on what points have occurred previously, and these might be different with every realization.

Fix some spatial interval, B. The integral of λ,

$A(t) = \int_0^t \iint_B \lambda(t,x,y)\, dxdy\, dt$, is called the *compensator* of *N*.

For any B, N(t, B) – A(t) is a *martingale*.

2. Conditional intensity, λ, continued.

Some authors (e.g. Jacod, 1975) actually define the compensator *A* as the unique predictable process such that N-A is a martingale, then define λ as a derivative of A, if it exists.

Jacod, J. (1975). Multivariate point processes: predictable projections, Radon-Nikodym derivatives, representation of martingales. Z. Wahrsch. verw. Geb. 31, 235–253.

We will use $\lambda(t,x,y) = \lim_{\Delta t,\, \delta \to 0} E(\, N([t,t+\Delta t) \times B(x,y,\delta)) \mid H_t\,) / (\Delta t\, \pi\delta^2)$.

λ is *predictable*.

Predictable technically is just a slight generalization of left-continuous. Technically it means adapted to the filtration generated by the left-continuous processes.

If for any (t,x,y), the limit doesn't exist or is ∞, we say λ doesn't exist.

λ uniquely determines the distributions of any simple point process.

2. Conditional intensity, λ, continued.

$\lambda(t,x,y) = \lim_{\Delta t,\, \delta \to 0} E(\, N([t,t+\Delta t) \times B(x,y,\delta)) \mid H_t\,) \,/\, (\Delta t\, \pi\delta^2)$.

λ uniquely determines the distributions of any simple point process.

Daley and Vere-Jones (2003), prop. 7.2.IV.

This is an amazing fact. For many stochastic processes, you need to know the conditional mean and variance and maybe more info in order to specify the process. For Gaussian processes, you need to specify the mean, variance, and covariance of the process. For simple point processes, all you need is the conditional mean, **λ**.

In modeling a point process, we typically assume it is simple and then just write down a model for **λ**.

3. Poisson processes.

The most basic models for point processes are the Poisson processes.

If N is a simple point process with conditional intensity $\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ does not depend on what points have occurred previously, then N is a *Poisson process*.

The name comes from the fact that for such a process, for any set B, N(B) has a Poisson distribution.
(I will assume throughout that B is measurable whenever we discuss N(B).)
$P(N(B) = k) = e^{-A} A^k / k!$,
for $k = 0, 1, 2, \ldots$,
where $A = \int_B \lambda(t,x,y)\, dtdxdy$,
and with the convention $0! = 1$.
The mean of N(B) is A and the variance is also A.

Siméon Poisson

Siméon Denis Poisson (1781–1840)

3. Poisson processes, continued.

A *Poisson process* is a simple point process with conditional intensity λ, where λ does not depend on what points have occurred previously.

Note that a Poisson process does not have to be stationary. A simple point process with $\lambda(t,x,y) = 10.5 + 2t + 4x - \exp(-y)$ is a Poisson process.

If λ is constant for all t,x,y, then N is a stationary Poisson process, and is sometimes called *completely random*.

Siméon Poisson

Siméon Denis Poisson (1781–1840)

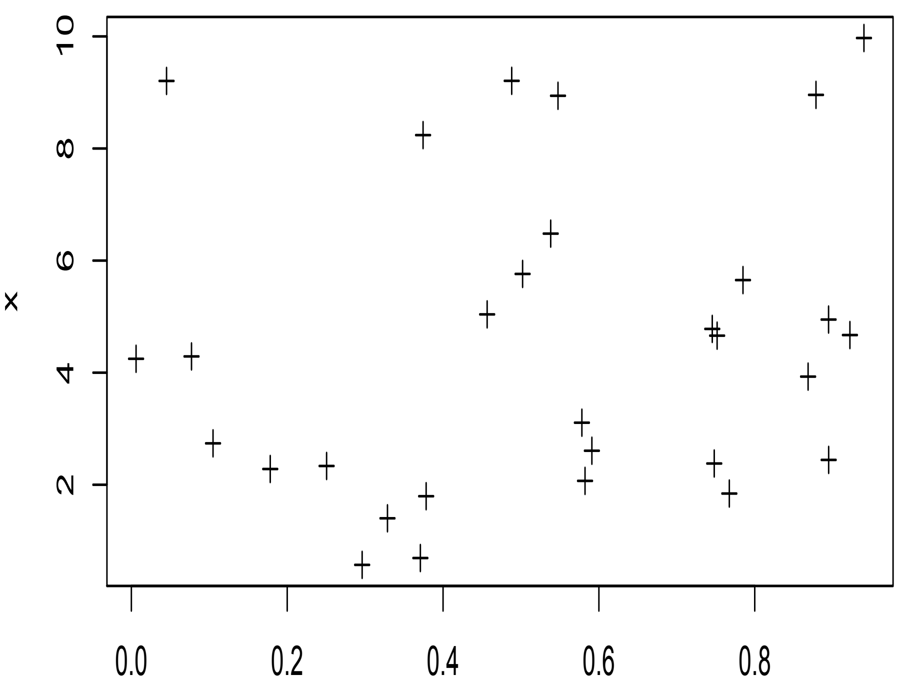
3. Poisson processes, continued.

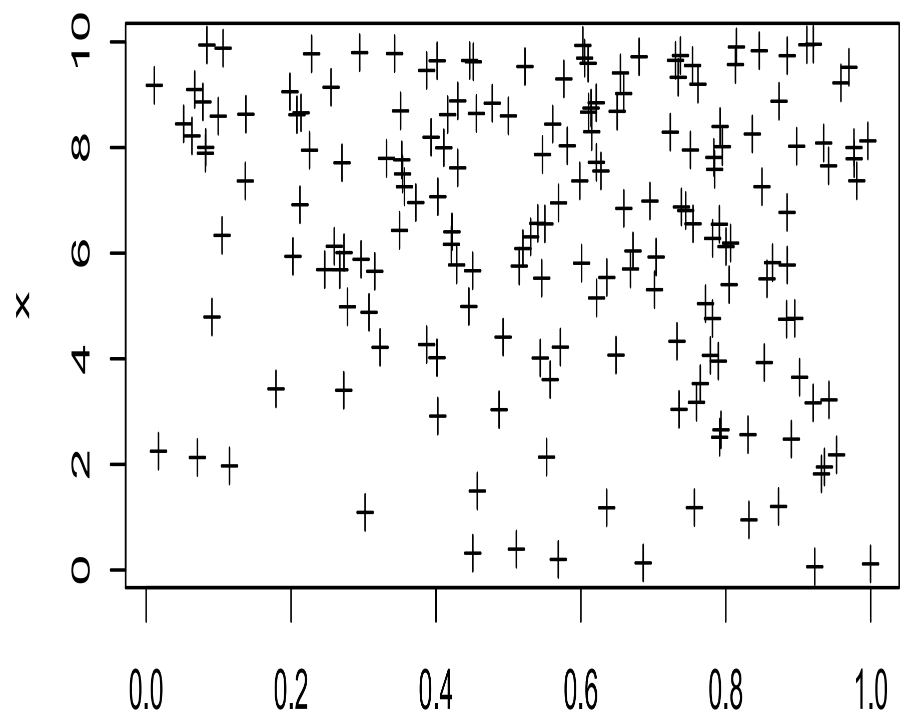
On the left is a stat. Poisson process with $\lambda(t,x) = 2.5$ on $[0,1] \times [0,10]$,

and on the right is a Poisson process with $\lambda(t,x) = 1.5 + 10t + 2x$.

The key thing about Poisson processes is their complete independence.

For a Poisson process N, $N(B_1)$ and $N(B_2)$ are independent for any disjoint sets B_1 and B_2.



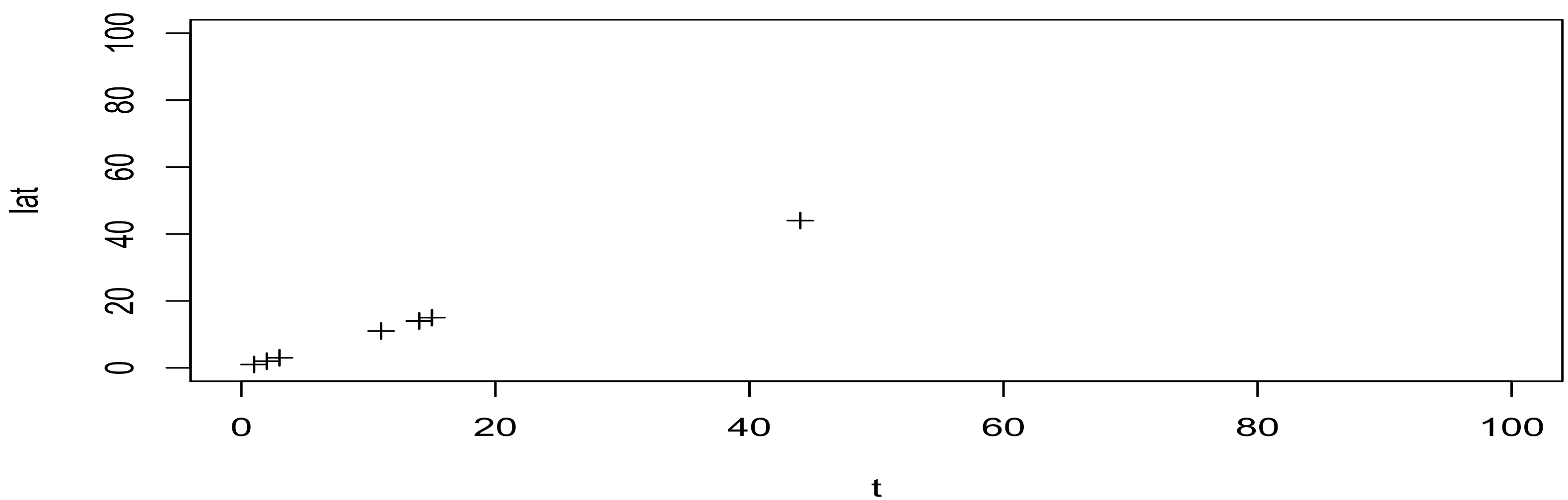


Exercises.

1. Suppose N is generated as follows. For each integer i = 1,2,..., N has a point at (i, i, i) with probability 1/i, independently of the other points, and N has no other points.

What is $\lambda(2,2,2)$?

a) 2. b) 1. c) ½. d) does not exist.



Exercises.

2. Suppose N is generated as follows. For each integer i = 1,2,..., N has a point at (i, i, i) with probability 1/i, independently of the other points, and N has no other points.

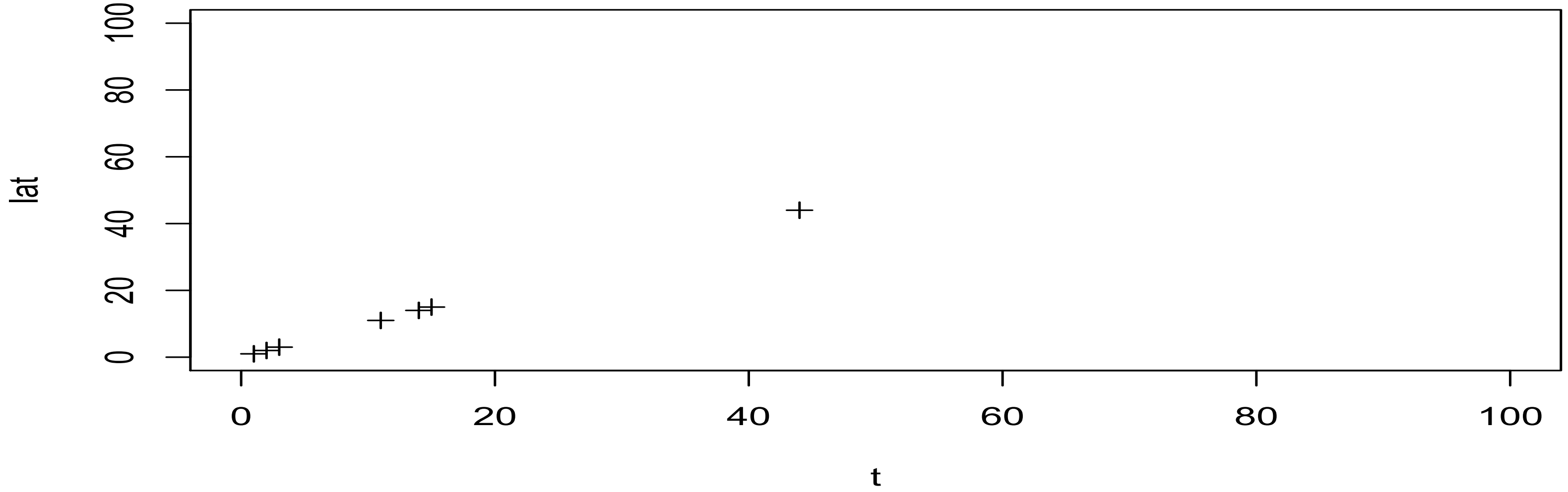
N is

a) simple but not orderly.

b) orderly but not simple.

c) simple and orderly.

d) neither simple nor orderly.

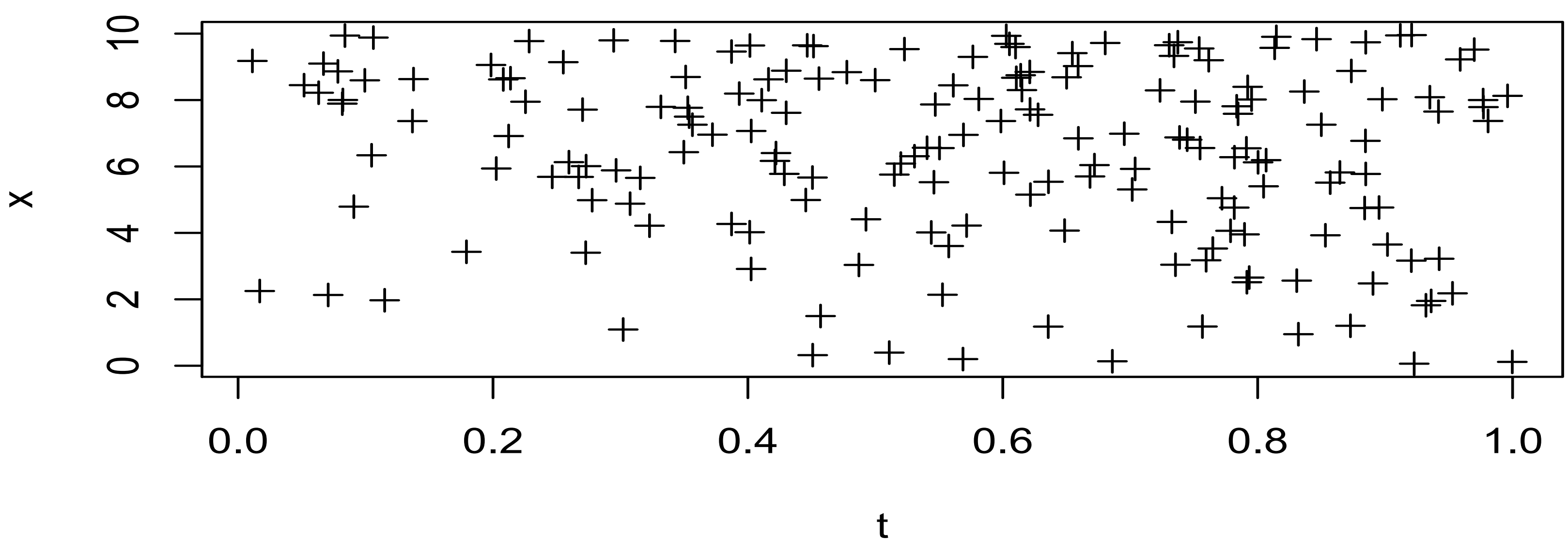


Exercises.

3. Suppose *N* is a Poisson process with $\lambda(t,x,y) = 1.5 + 10t + 2x$

on $B = [0,1] \times [0,10] \times [0,1]$.

What is $EN(B)$?



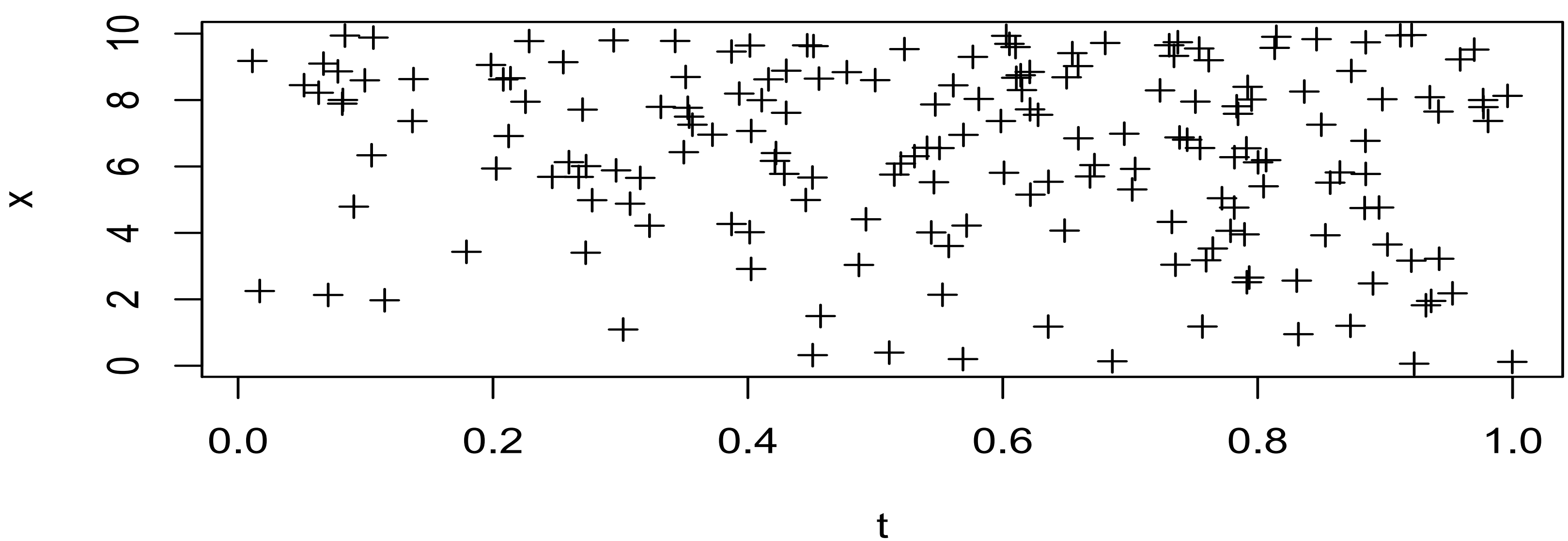
Exercises.

3. Suppose N is a Poisson process with $\lambda(t,x,y) = 1.5 + 10t + 2x$

on B = [0,1] x [0,10] x [0,1].

What is EN(B)?

$\int_B \lambda(t,x,y)\, dt\, dx\, dy = 1.5(10) + 10(10)(1^2)/2 + 2(1)(10^2)/2 =$
$15+50+100=165.$



Code.

```
## nonsimple point process
n = 20
x = runif(n)
y = runif(n)
plot(x,y,xlab="t",ylab="lat",pch=2)
points(x[20],y[20],pch=3)

## nonsimple ground process
plot(x,y,xlab="t",ylab="lat",pch=2)
points(x[20],y[20]+.05,pch=3)

## nonorderly process
plot(c(0,1),c(0,1),type="n",xlab="t",ylab="lat")
n = 100
for(i in 1:n) points(1/i,runif(1),pch=3,cex=.5)
```

Code.

```
## points at (i,i) with prob. 1/i.
plot(c(0,100),c(0,100),type="n",xlab="t",ylab="lat")
for(i in 1:100) if(runif(1) < 1/i) points(i,i,pch=3)

## stationary Poisson process with intensity 2.5 on B=[0,1]x[0,10].
n = rpois(1,2.5*1*10)
t = runif(n)
x = runif(n)*10
plot(t,x,pch=3)
```

Code.

```
## nonstationary Poisson process with intensity 1.5+10t+2x on B.
n = rpois(1,15+50+100)
n1 = 0
t = c()
x = c()
while(n1<n){
t2 = runif(1) ## candidate point
x2 = runif(1)*10
if(runif(1) < (1.5+10*t2+2*x2)/(1.5+10+20)){ ## keep it
  t = c(t,t2)
  x = c(x,x2)
  n1 = n1 + 1
  cat(n1," ")
}
}
plot(t,x,pch=3)
```