Statistics 222, Spatial Statistics.

Outline for the day:

1. Problems and code from last lecture.
2. Likelihood.
3. MLE.
4. Simulation.

1. Questions and code from last time.

The difference between ETAS and a Hawkes process is ...

a) an ETAS process is more strongly clustered.
b) the points of an ETAS process all occur at different locations.
c) the points of an ETAS process all have different productivity.
d) the points of an ETAS process all have different triggering functions.

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Which of the following can possibly have two points within distance .01 of each other?

a) a hardcore process with $\sigma = .01$.
b) a Strauss process with $R = .01$.
c) a Matern I process with $r = .01$.
d) a Matern II process with $r = .01$.

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Code.

```
install.packages("spatstat")
library(spatstat)

## STRAUSS process
z = rStrauss(100,0.7,0.05)
plot(z, pch=2,cex=.5)

## HARDCORE process
z = rHardcore(100,0.05)
plot(z, pch=2,cex=.5)

## MATERN(I).
z = rMaternI(20,.05)
plot(z, pch=2,cex=.5)
```

Code.

```
## MATERN(II)
z = rMaternII(100,.05)
plot(z,pch=2,cex=.5)

## HAWKES.
install.packages("hawkes")
library(hawkes)
lambda0 = c(0.2,0.2)
alpha = matrix(c(0.5,0,0,0.5),byrow=TRUE,nrow=2)
beta = c(0.7,0.7)
horizon = 3600
h = simulateHawkes(lambda0,alpha,beta,horizon)
plot(c(0,3600),c(0,3),type="n",xlab="t",ylab="x")
points(h[[1]],.5+runif(length(h[[1]])),pch=2,cex=.1)
points(h[[2]],1.5+runif(length(h[[2]])),pch=3,cex=.1)
```

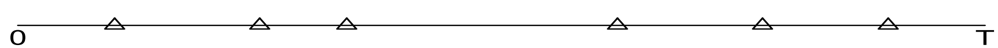
2. Likelihood, continued.

For iid draws $t_1, t_2, \ldots, t_n$, from some density $f(\theta)$, the likelihood is simply
$L(\theta) = f(t_1 ; \theta) \times f(t_2 ; \theta) \times \ldots \times f(t_n ; \theta) = \prod f(t_i ; \theta)$.
This is the probability density of observing $\{t_1, t_2, \ldots, t_n\}$, as a function of θ.

For a Poisson process with intensity $\lambda(\theta)$ on $[0,T]$, the likelihood of observing the points $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is simply

$$\lambda(\tau_1) \times \lambda(\tau_2) \times \ldots \times \lambda(\tau_n) \times \exp\{-A(\tau_1)\} \times \exp\{-(A(\tau_2)-A(\tau_1))\} \times \ldots \times \exp\{-(A(T)-A(\tau_n))\},$$

$$= \prod \lambda(\tau_i) \exp\{-A(T)\},$$



where $A(t) = \int_0^t \lambda(u)du$.
$P\{k \text{ points in } (\tau_1, \tau_2)\}$ is $\exp(-B)\, B^k/k! = \exp(-B)$ for $k = 0$,
where $B = \int_{\tau 1}^{\tau 2} \lambda(t)dt$.

So the log likelihood is $\sum \log(\lambda(\tau_i)) - A(T)$.
In the spatial-temporal case, the log likelihood is essentially the same,
$\sum \log(\lambda(\tau_i)) - \int \lambda(t,x,y)\, dt\, dx\, dy$.

3. Maximum likelihood estimation.

Find $\hat{\theta}$ (= θˆ) maximizing $l(\theta) = \sum \log(\lambda(\tau_i)) - \int \lambda(t,x,y)\, dt\, dx\, dy$.

Ogata (1978) showed that the resulting estimate, θˆ, is, under standard conditions,

asymptotically unbiased, $E(\hat{\theta}) \rightarrow \theta$,

consistent, $P(|\hat{\theta} - \theta| > \varepsilon) \rightarrow 0$ as $T \rightarrow \infty$, for any $\varepsilon > 0$,

asymptotically normal, $\hat{\theta} \rightarrow_D$ Normal as $T \rightarrow \infty$,

and asymptotically efficient, min. variance anong asymptotically unbiased estimators.

Further, he showed standard errors for θˆ can be constructed using the diagonal elements of the inverse of the Hessian of L evaluated at θˆ .

sqrt(diag(solve(loglikelihood$hess)))

Ogata, Y. (1978). The asymptotic behaviour of maximum likelihood estimators for stationary point processes. Ann. Inst. Statist. Math. 30, 243-261.

3. Maximum likelihood estimation continued.

The conditions of Ogata (1978) can be relaxed a bit for Poisson processes [1], and for certain spatial-temporal process in general [2].

Even if the process is not Poisson, under some circumstances [3] the parameters governing the unconditional intensity, $E\lambda$, can be consistently estimated by maximizing $L_P(\theta) = \sum \log(E\lambda(\tau_i)) - \int E\lambda(t,x,y)\, dt\, dx\, dy$. Basically pretend the process is Poisson.

Suppose you are missing some covariate that might affect λ. Under general conditions, the MLE will nevertheless be consistent, provided the effect of the missing covariate is small [4].

[1] Rathbun, S.L., and Cressie, N. (1994). Asymptotic properties of estimators for the parameters of spatial inhomogeneous Poisson point processes. Adv. Appl. Probab. 26, 122–154.

[2] Rathbun, S.L., (1996). Asymptotic properties of the maximum likelihood estimator for spatio-temporal point processes. *JSPI* 51, 55–74.

[3] Schoenberg, F.P. (2004). Consistent parametric estimation of the intensity of a spatial-temporal point process. *JSPI* 128(1), 79--93.

[4] Schoenberg, F.P. (2016). A note on the consistent estimation of spatial-temporal point process parameters. *Statistica Sinica,* 26, 861-879.

3. Maximum likelihood estimation continued.

λ is completely separable if $\lambda(t,x,y;\theta) = \theta_3\, \lambda_0(t;\theta_0)\, \lambda_1(t, x; \theta_1)\, \lambda_2(t, y; \theta_2)$.
Suppose N has marks too. λ is separable in mark (or coordinate) *i* if
$\lambda(t, x, y, m_1, m_2, \ldots, m_k ; \theta) = \theta_2\, \lambda_i(t, m_i; \theta_i)\, \lambda_{-i}(t, x, y, m_{-i} ; \theta_{-i})$.

Suppose you are neglecting some *mark* or coordinate of the process. Under some conditions, the MLE of the other parameters will nevertheless be consistent [1].

In maximizing $L(\theta) = \sum \log(\lambda(\tau_i)) - \int \lambda(t,x,y)\, dt\, dx\, dy$,
it is typically straightforward to compute the sum, but the integral can be tricky esp. when the conditional intensity is very volatile. One trick noted in [2] is that, for a Hawkes process where $\lambda(t,x,y) = \mu(x,y) + \kappa \sum_{\{t',x',y':\ t' < t\}} g(t-t',x-x',y-y')$, where g is a density, and $\int\mu(x,y)dxdy = \mu$,

$$\begin{aligned}\int \lambda(t,x,y)\, dt\, dx\, dy &= \mu T + \kappa \int \sum g(t-t',x-x',y-y')\, dt\, dx\, dy \\ &= \mu T + \kappa \sum \int g(t-t',x-x',y-y')\, dt\, dx\, dy \\ &\sim \mu T + \kappa N.\end{aligned}$$

[1] Schoenberg, F.P. (2016). A note on the consistent estimation of spatial-temporal point process parameters. *Statistica Sinica,* 26, 861-879.
[2] Schoenberg, F.P. (2013). Facilitated estimation of ETAS. *Bulletin of the Seismological Society of America,* 103(1), 601-605.

4. Simulation.

Jesper Møller

One can simulate spatial-temporal point processes by *thinning*.
Lewis, P. and Shedler, G. (1979). Simulation of nonhomogeneous poisson processes by thinning. *Naval Research Logistics Quarterly*, 26:403–413, 1979.

Suppose λ has some upper bound, B. $\lambda(t,x,y) \leq B$ everywhere.
First, simulate a stationary Poisson process N with intensity B.
For i = 1,2, ... keep point τ_i with probability $\lambda(\tau_i)/B$. We saw this in day 4 for simulating inhomogeneous Poisson processes, but it works for other processes too.

Boundary issues can be important in simulation. For Gibbs processes, for instance, the simulation can be biased because of missing points outside the observation region. For Hawkes processes, the simulation will tend to be biased by having too few points at the beginning of the simulation. One can have burn-in, by simulating points outside the observation region or before time 0, or in some cases some fancy weighting schemes can be done to achieve *perfect* simulation without burn-in. See Møller, J. and Waagepetersen, R. (2003). *Statistical Inference and Simulation for Spatial Point Processes*. Chapman and Hall, Boca Raton.

5. More exercises.

a. Suppose N is a Poisson process with intensity $\lambda(t,x,y) = \exp(3t)$ over t in [0,10], x in [0,1], y in [0,1].

N happens to have points at (1.5, .4, .2)
(2, .52, .31)
(4, .1, .33)
(5, .71, .29).

What is the log-likelihood of this realization?

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(2, .52, .31)
(4, .1, .33)
(5, .71, .29).

What is the log-likelihood of this realization?

$-4.5-6-12-15 - \iiint \exp(-3t)\, dt\, dx\, dy$

$= -37.5 - \int_0^{10} \exp(-3t)\, dt$, because x and y go from 0 to 1,

$= -37.5 - \exp(-3t) / (-3)\Big]_0^{10}$

$= -37.5 + \exp(-30)/3 - \exp(0)/3$

$= -37.5 + \exp(-30)/3 – 1/3$

~ -37.83.

5. More exercises.

Which of the following is not typically true of the MLE of a spatial-temporal point process?

a. It is unbiased.
b. It is consistent.
c. It is asymptotically normal.
d. It is asymptotically efficient.

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Entering data example.

```
## First, input 54 points using the mouse.
n = 54
plot(c(0,1),c(0,1),type="n",xlab="longitude",ylab="latitude",
        main="locations")
 x1 = rep(0,n)
 y1 = rep(0,n)
 for(i in 1:n){
 z1 = locator(1)
 x1[i] = z1$x
 y1[i] = z1$y
 points(x1[i],y1[i])
 }
```

```
##### PLOT THE POINTS WITH A 2D KERNEL SMOOTHING IN
     GREYSCALE PLUS A LEGEND
library(splancs)
bdw = sqrt(bw.nrd0(x1)^2+bw.nrd0(y1)^2)  ## possible default bandwidth
b1 = as.points(x1,y1)
bdry = matrix(c(0,0,1,0,1,1,0,1,0,0),ncol=2,byrow=T)
z = kernel2d(b1,bdry,bdw)
attributes(z)
par(mfrow=c(1,2))
image(z,col=gray((64:20)/64),xlab="km E of origin",ylab="km N of
     origin")
points(b1)
x4 = seq(min(z$z),max(z$z),length=100)
```

```
plot(c(0,10),c(.8*min(x4),1.2*max(x4)),type="n",
        axes=F,xlab="",ylab="")
image(c(1:1),x4,matrix(rep(x4,2),ncol=100,byrow=T),
        add=T,col=gray((64:20)/64))
text(2,min(x4),as.character(signif(min(x4),2)),cex=1)
text(2,(max(x4)+min(x4))/2,
        as.character(signif((max(x4)+min(x4))/2,2)),cex=1)
text(2,max(x4),as.character(signif(max(x4),2)),cex=1)
mtext(s=3,l=-3,at=1,"density (pts/km^2)")
```

We will continue this next time, fitting models to this by MLE.