Statistics 222, Spatial Statistics.

Outline for the day:

1. Correction on day08.r.
2. Problems.
3. Application of nonparametric estimates to earthquakes and plague.
4. Simulating Hawkes processes.
5. Estimating Hawkes processes.

1. Correction in day08.r.

I had this.

Fitting a Pseudo-Likelihood model.
I'm using the model lambda_p (z | z_1, ..., z_k) =
mu + alpha x + beta y + gamma SUM_{i = 1 to k} a1 exp{-a1 D(z_i,z)}
where z = (x,y), and where D means distance.
So, if gamma is positive, then there is clustering; otherwise inhibition.

But $g(r) = a_1 \exp(-a_1 r)$ is actually not a density.
$g(t) = a_1 \exp(-a_1 t)$ is a density, because $\int_0^\infty a_1 \exp(-a_1 t)\, dt = 1$, for $a_1 > 0$,
but not $\int\int a_1 \exp(-a_1 r)\, dx\, dy$.

$a_1 \exp(-a_1 r) / (2\pi r)$ is a spatial density, because
$\iint a_1 \exp(-a_1 r) / (2\pi r)\, dx\, dy = \int_0^{2\pi} \int_0^\infty a_1 \exp(-a_1 r) / (2\pi r)\, r\, dr\, d\phi$
$= \int_0^\infty a_1 \exp(-a_1 r)\, dr$
$= 1$.

So I should have fit **lambda_p (z | z_1, ..., z_k) =**
mu + alpha x + beta y + gamma SUM_{i = 1 to k} a1/2π exp{-a1 D(z_i,z)}/D(z_i,z).
I corrected this in the current version of day08.r.

2. Problems.

Suppose you observe a Poisson process with rate μ on the space-time window
[0,1]x[0,1] x [0,10], and it happens to have 5 points.
S T.
What is the log-likelihood, $l(\mu)$?

a) $5 \mu + 10 \exp(\mu)$.
b) $5 \log(\mu) - 10 \mu$.
c) $5 + 10 \log(\mu)$.
d) $5 \exp(\mu) + 5 \log(\mu)$.

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Problems.

Suppose you observe a Poisson process with rate 3t on the space-time window
[0,1]x[0,1] x [0,10].
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How many points do you expect to observe?

a) 50.
b) 100.
c) 150.
d) 200.

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$\int\int\int 3t \, dx \, dy \, dt = \int 3t \, dt = 3t^2/2 \,]_0^{10} = 300/2 - 0 = 150.$

Problems.

Suppose you observe a Hawkes process with conditional intensity
$\lambda(t,x,y) = 2 + 0.6 \int f(t-t')\, g(x-x',y-y')\, dN(t',x',y')$, on the space-time window
$\underbrace{[0,1]\mathrm{x}[0,1]}_{S}$ x $\underbrace{[0,10]}_{T}$,

where $f(t)$ is a density like $f(t) = 4\exp(-4t)$, and $g(x,y)$ is a planar density like
$g(x,y) = 3 \exp(-3r) / (2\pi\, r)$, where $r = \sqrt{(x^2+y^2)}$.

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$20 + 20 \times .6 + 20 \times .6^2 + 20 \times .6^3 + \ldots = 20/(1-.6) = 20/.4 = 50.$

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$g(x,y) = 3 \exp(-3r) / (2\pi r)$, where $r = \sqrt{(x^2+y^2)}$.

You observe 2 points, at $(t,x,y) = (1,.5,.5)$ and $(3,.5,.6)$. What is the log-likelihood?

a) $\log(2) + \log(2 + 36 \exp(-8.3)/\pi) - 21.2$.
b) $\log(3.2) + \log(2 + 36 \exp(-8.3)/\pi) - 20$.
c) $\log(3.2) + \log(2 + 36 \exp(-8.3)/\pi) - 20$.
d) $2\log(2) - 20$.

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c) $\log(3.2) + \log(2 + 36 \exp(-8.3)/\pi) - 20$.
d) $2\log(2) - 20$.
$\sum\log(\lambda) - \int\lambda d\mu = \log(2) + \log\{2 + .6(4\exp(-8))(3\exp(-.3)/(.2\pi))\} - 20 - .6 - .6$
$= \log(2) + \log(2 + 36 \exp(-8.3)/\pi) - 21.2$.

3. Nonparametric Marsan and Lengliné (2008) estimator.

Marsan and Lengliné (2008) assume g is a step function, and estimate steps β_k as parameters.

$$\ell(\theta) = \sum_i \log\left(\lambda(\tau_i, \mathbf{x}_i | \mathcal{H}_{\tau_i})\right) - \int_0^T \int_S \lambda(t, \mathbf{x} | \mathcal{H}_t)\, d\mathbf{x} dt$$

Setting the partial derivatives of this loglikelihood with respect to the steps β_k to zero yields

$$0 = \partial\ell(\theta)/\partial\beta_k = \sum_{(i,j):\tau_i - \tau_j \in U_k} K/\lambda(\tau_i) - Kn|U_k|,$$

where $|U_k|$ is the width of step k, for $k = 1,2,\ldots, p$. This is a system of p equations in p unknowns.

However, the equations are nonlinear. They depend on $1/\lambda(\tau_i)$.

Gradient descent methods: way too slow for large p.

Marsan and Lengliné (2008) find *approximate* maximum likelihood estimates using the E-M method for point processes. You pick initial values of the parameters, then given those, you know the probability event i triggered event j. Using these, you can weight each pair of points by its probability and re-estimate the parameters, and repeat until convergence.

This method works well but is iterative and time-consuming.

Nonparametric Marsan and Lengliné (2008) estimator.

The last step of Marsan and Lengliné uses essentially a histogram estimator.

Others have used slightly different approaches for smoothing.

Lewis and Mohler (2011) use maximum penalized likelihood.

Bacry et al. (2012) use the Laplace transform of the covariance function.

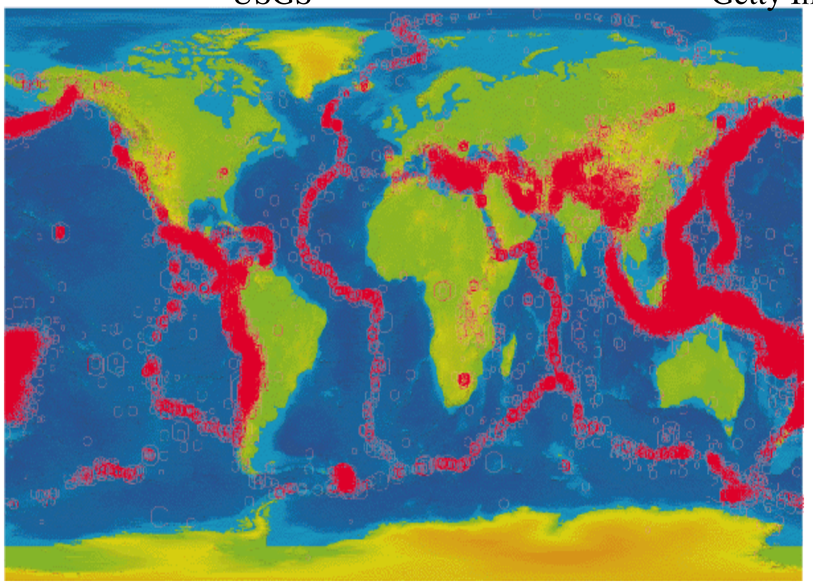
Adelfio and Chiodi (2015) use a semi-parametric method where the background rate λ is estimated nonparametrically and the triggering function g parametrically.

There are also standard non-parametric methods for smoothing points generally, using splines, kernels, or wavelets. (Brillinger 1997).

Marsan and Lengliné's method is different. It estimates g, not the overall rate.

Applications to earthquakes and US plague.

USGS



Getty Images

Application to Loma Prieta earthquake data.

Loma Prieta earthquake was Mw 6.9 on Oct 17, 1989.

As an illustration, we will estimate g on its 5566 aftershocks $M \geq 3$ within 15 months.

(Google images)

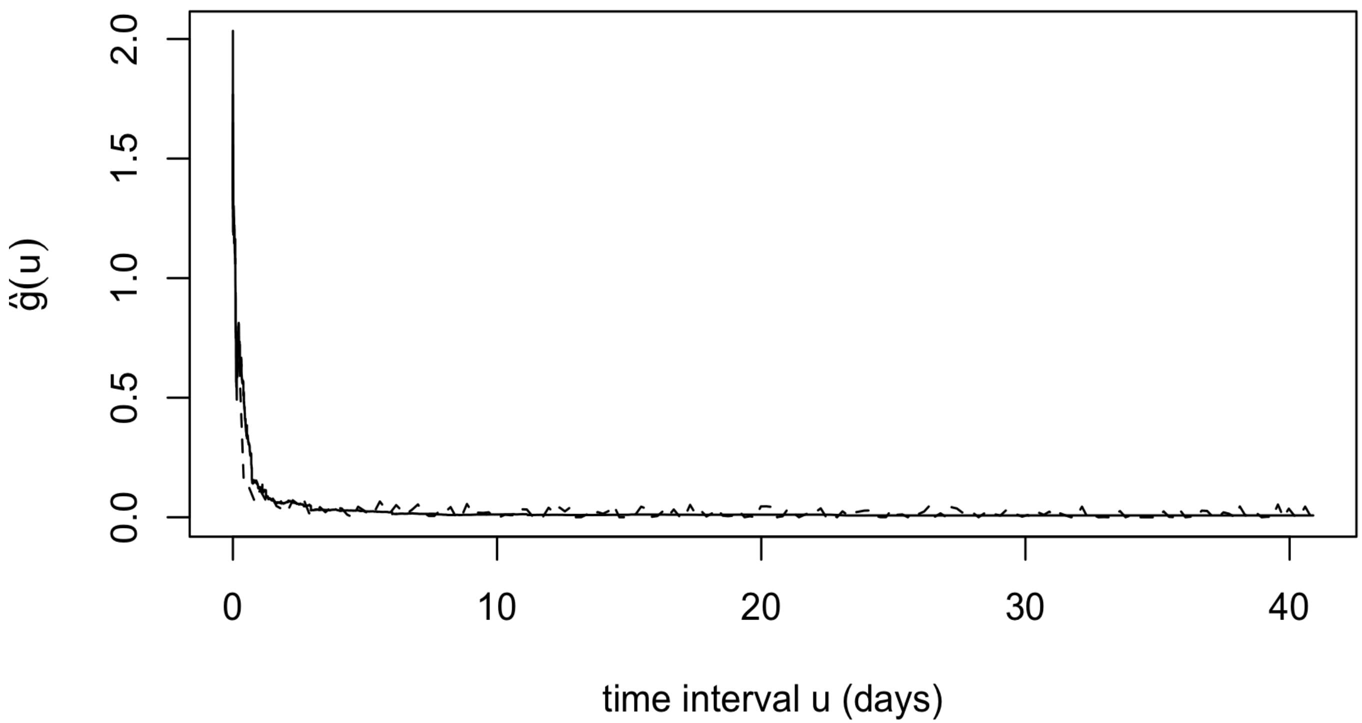
Application to Loma Prieta earthquake data. (SCEC.ORG)

Estimated triggering function for 5567 Loma Prieta M ≥ 3 events, 10/16/1989 to 1/17/1990.

Solid curve is the analytic method and dashed curve is Marsan and Lengliné (2008).

Dotted curves are estimates based on analytic method +/- 1 or 2 SEs, respectively, for light grey and dark grey.

SEs were computed using the SD of analytic estimates in 100 simulations of Hawkes processes with triggering functions sampled from the solid curve.



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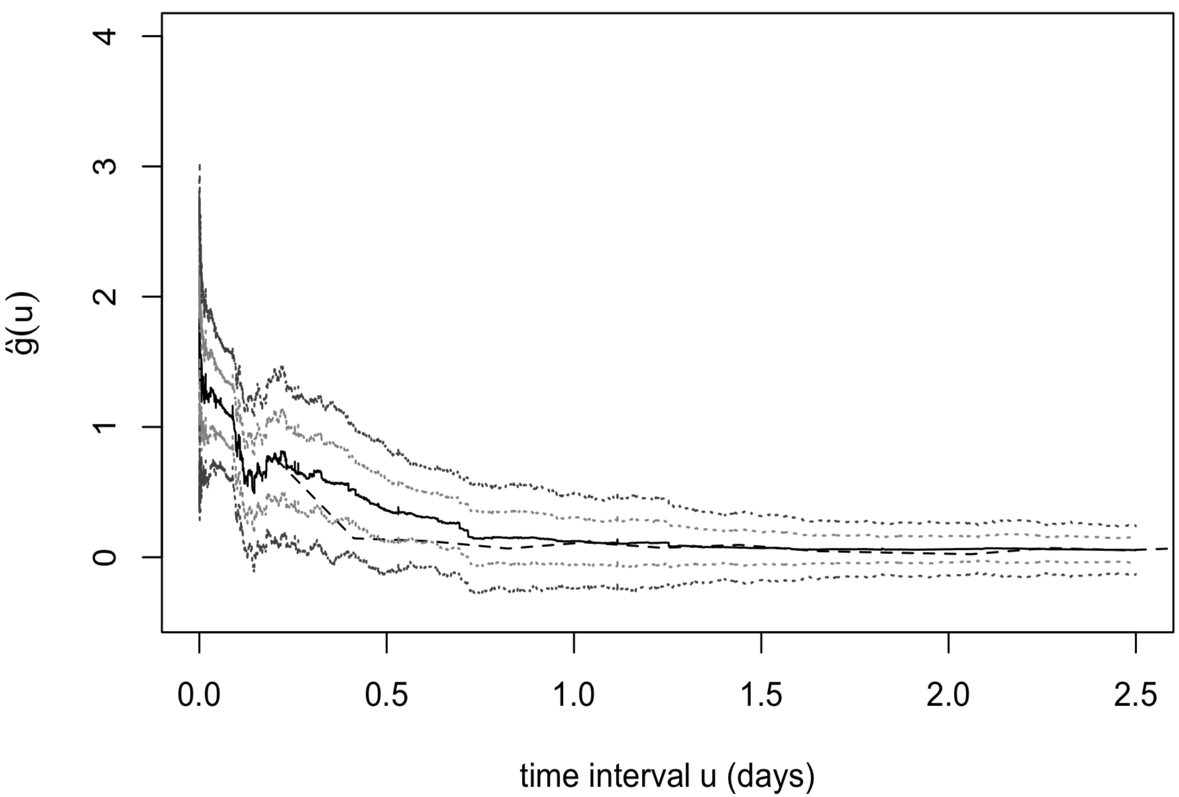

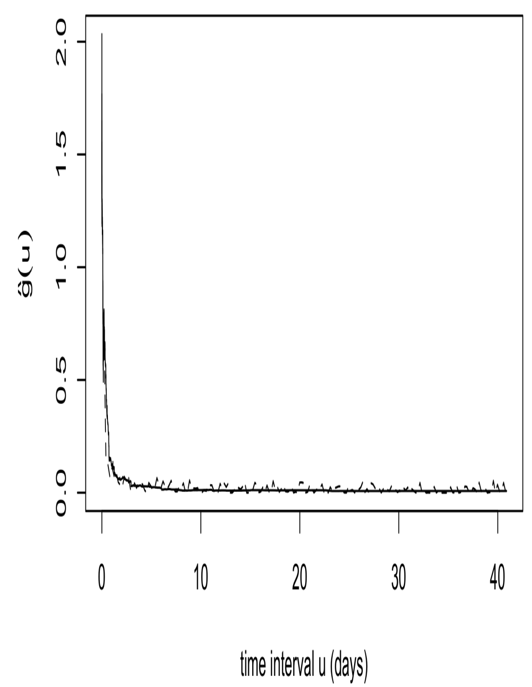

Application to US plague data.

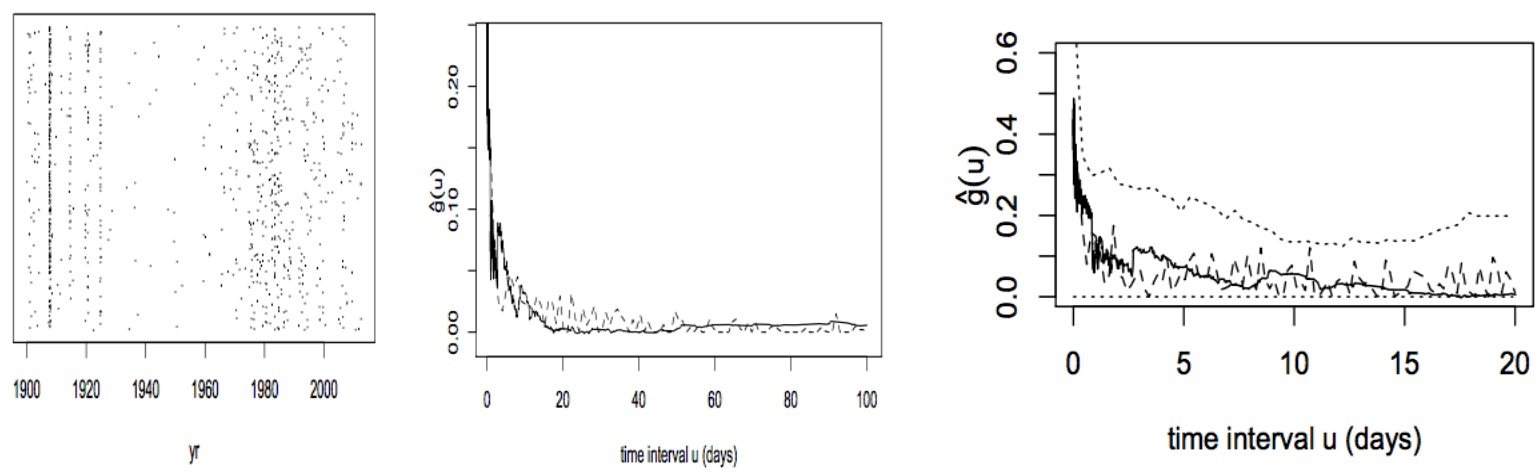

Figure 4: (a) Onset dates of reported and confirmed occurrences of plague in the United States from 1900-2012, according to data from the CDC. The y-coordinates are scattered uniformly at random on the y-axis for ease of visualization. (b) Estimated triggering function, $\hat{g}$, for the reported onset times of U.S. plague cases. (c) Estimated triggering function $\hat{g}$, for U.S. plague data, for intervals up to 20 days. In (b) and (c), the solid curves correspond to equation (9), the dashed curves result from the method of Marsan and Lengliné (2008), and the dotted curves are the middle 95% range for $\hat{g}$ from equation (9) resulting from simulating Hawkes models where the true triggering function is that estimated from the data using equation (9).