# Statistics 222, Spatial Statistics.

Outline for the day:

1. Integral term in loglikelihood for Hawkes processes.
2. Estimating Hawkes processes using MLE.
3. Nonparametric estimation.

1. The integral term in the loglikelihood for Hawkes processes.

loglikelihood = $\sum_i \log(\lambda(t_i,x_i,y_i)) - \iiint \lambda(t,x,y)\, dxdydt$.

The space-time region is B = [0,T] x S.

For a Hawkes process, $\lambda(t,x,y) = \mu\, \rho(x,y) + K\sum_{i:ti<t} g(t-t_i, x-x_i, y-y_i)$, where $\rho$ and g are densities.

$\int_0^T \iint \lambda(t,x,y)\, dxdydt = \int_0^T \iint \mu\, \rho(x,y)\, dxdydt + \int_0^T \iint K\sum_{i:ti<t} g(t-t_i, x-x_i, y-y_i)\, dxdydt$

$= \mu\, T + \int_0^T \iint K\int_B 1_{\{t'<t\}}\, g(t-t', x-x', y-y')\, dN(t',x',y')\, dxdydt$

interchanging the integrals

$= \mu\, T + K\int_B \int_0^T \iint 1_{\{t'<t\}}\, g(t-t', x-x', y-y')\, dxdydt\, dN(t',x',y')$

changing coordinates, letting u = t-t', v = x-x', w = y-y',

$= \mu\, T + K\int_B \int_0^{T-t'} \iint_{S-(x',y')} g(u, v, w)\, dudvdw\, dN(t',x',y')$

$\sim \mu\, T + K \int_B (1)\, dN(t',x',y')$

$= \mu\, T + KN(B)$.

This is approximate because typically $\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty g(u,v,w)\, dudvdw = 1$, but instead, we have

$\int_0^{T-t'} \iint_{S-(x',y')} g(u,v,w)\, dudvdw$ which is often close to 1.

**Nonparametric Marsan and Lengliné (2008) estimator.**

Marsan and Lengliné (2008) assume g is a step function, and estimate steps $\beta_k$ as parameters.

$$\ell(\theta) = \sum_i \log\left(\lambda(\tau_i, \mathbf{x}_i | \mathcal{H}_{\tau_i})\right) - \int_0^T \int_S \lambda(t, \mathbf{x} | \mathcal{H}_t)\, d\mathbf{x}dt$$

Setting the partial derivatives of this loglikelihood with respect to the steps $\beta_k$ to zero yields

$$0 = \partial\ell(\theta)/\partial\beta_k = \sum_{(i,j):\tau_i - \tau_j \in U_k} K/\lambda(\tau_i) - Kn|U_k|,$$

where $|U_k|$ is the width of step k, for $k = 1,2,..., p$. This is a system of $p$ equations in $p$ unknowns.

However, the equations are nonlinear. They depend on $1/\lambda(\tau_i)$.

Gradient descent methods: way too slow for large $p$.

Marsan and Lengliné (2008) find *approximate* maximum likelihood estimates using the E-M method for point processes. You pick initial values of the parameters, then given those, you know the probability event $i$ triggered event $j$. Using these, you can weight each pair of points by its probability and re-estimate the parameters, and repeat until convergence.

This method works well but is iterative and time-consuming.

**Analytic solution.**

Set $p = n$. (*$p$ = number of steps in the step function, $g$, and $n$ = # of observed points.*)

Setting the derivatives of the loglikelihood to zero we have the $p$ equations

$$0 = \partial \ell(\theta)/\partial \beta_k = \sum_{(i,j):\tau_i - \tau_j \in U_k} K/\lambda(\tau_i) - Kn|U_k|,$$

which are $p$ linear equations in terms of $1/\lambda(\tau_i)$, for $i = 1,2, ..., n$. (!)

So, if $p=n$, then we can use these equations to solve for $1/\lambda(\tau_i)$,

and if we know $1/\lambda(\tau_i)$, then we know $\lambda(\tau_i)$,

and if we know $\lambda(\tau_i)$, then we can solve for $\beta_i$ because the def. of a Hawkes process is

$$\lambda(\tau_j) = \mu + K \sum_{i<j} g(\tau_j - \tau_i),$$

which results in $n$ linear equations in the $p$ unknowns $\beta_1, \beta_2, ..., \beta_p$, when $g$ is a step function.

**Analytic solution.**

We can write the resulting estimator in very condensed form.

Let $\boldsymbol{\lambda} = \{\lambda(\tau_1),\lambda(\tau_2),\ldots, \lambda(\tau_n)\}$.

Suppose the steps of $g$ have equal widths, $|U_1| = |U_2|$, etc. Call this width U.

Let $A_{ij}$ = the number points $\tau_k$ such that $\tau_j - \tau_k$ is in $U_i$, for $i,j$ in $\{1,2,\ldots,p\}$.

Then the loglikelihood derivatives equalling zero can be rewritten

$$0 = KA(1/\boldsymbol{\lambda}) - Kb,$$

where $\mathbf{b} = nU\mathbf{1}$, with $\mathbf{1} = \{1,1,\ldots,1\}$.

This has solution $\underline{1/\boldsymbol{\lambda} = A^{-1} b}$, if A is invertible.

Similarly, the Hawkes equation can be rewritten $\boldsymbol{\lambda} = \mu + KA^T\boldsymbol{\beta}$, whose solution is

$$\underline{\hat{b} = (KA^T)^{-1}(\lambda-\mu)}.$$

Combining these two underlined formulas yields the estimates

$$\hat{\beta} = (KA^T)^{-1}[1/(A^{-1}b) - \mu]$$

This is very simple, trivial to program, and rapid to compute.

**Analytic solution.**

There are problems, however.

1. Estimating $n$=$p$ steps. High variance.

    However, if we can assume $g$ is smooth, then we can smooth our estimates for stability.

2. Need to estimate $K$ and $\mu$ too.

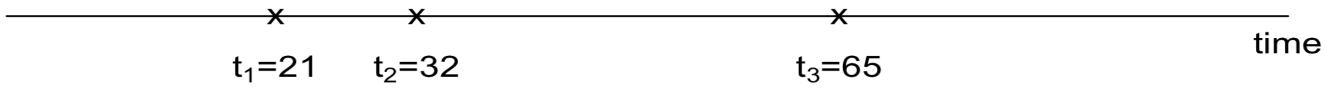
    We can use Marsan and Lengliné's method or take derivatives for these as well.

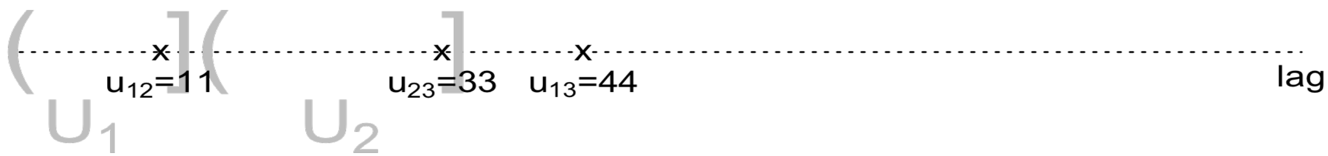
3. What about spatially-varying steps and unequally sized steps for $g$?

    No problem. The estimation generalizes in a completely obvious way.

4. $A$ can be singular.
    We may need better solutions for this.
    I let $u_j = \tau_j - \tau_{j-1}$ , sorted the $u_j$ values, and then used $[u_{(1)}, u_{(2)})$, etc. as my binwidths, so each row and column of $A$ would have at least one non-zero entry. If it still isn't singular, adding in a few random 1's into $A$ often helps.

$$\hat{\beta} = (KA^T)^{-1}[1/(A^{-1}b) - \mu]$$

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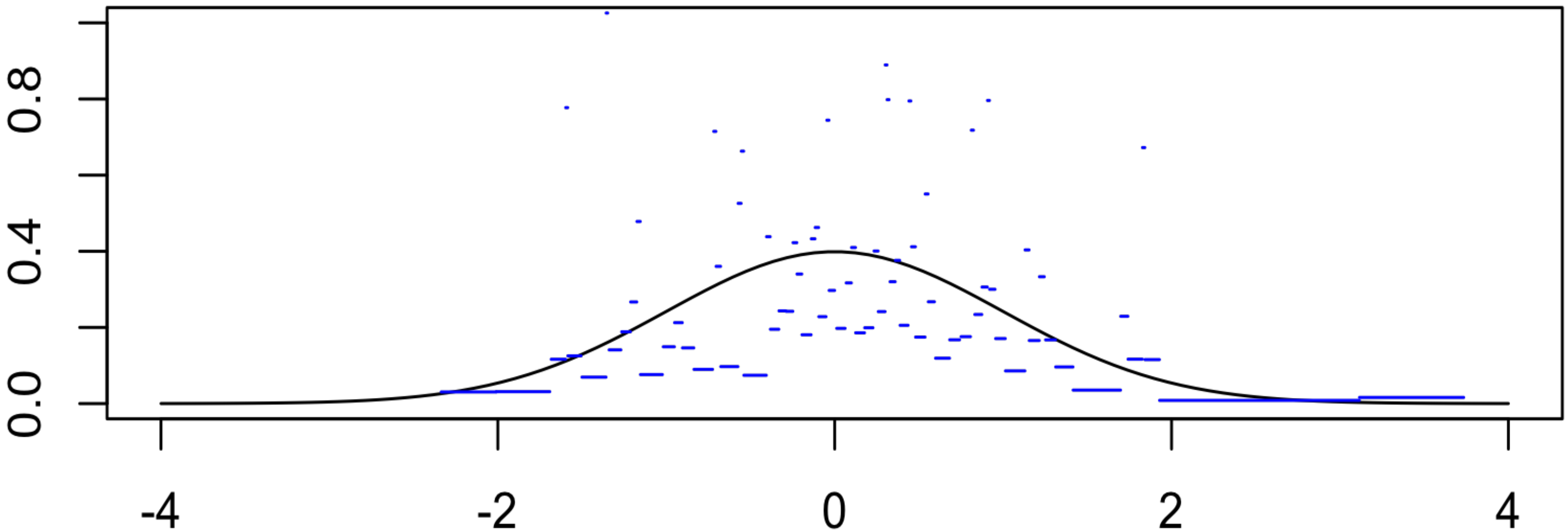
$$\hat{\beta} = (KA^T)^{-1}[1/(A^{-1}b) - \mu]$$

Note: take the simple case of a dataset where point i is only influenced by point i-1. This is basically a renewal process, and we are just estimating a renewal density.

Here $A = I$, $K = 1$, and we get the density estimator $1/\{n(x_i - x_{i-1})\}$.

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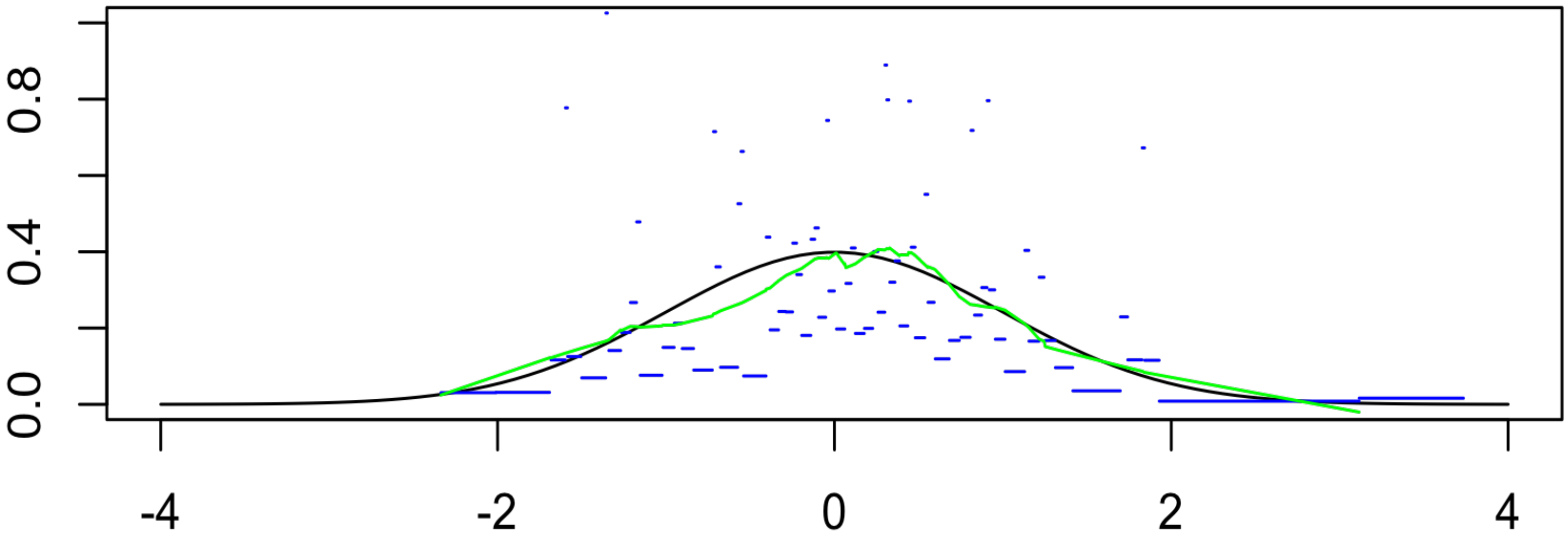


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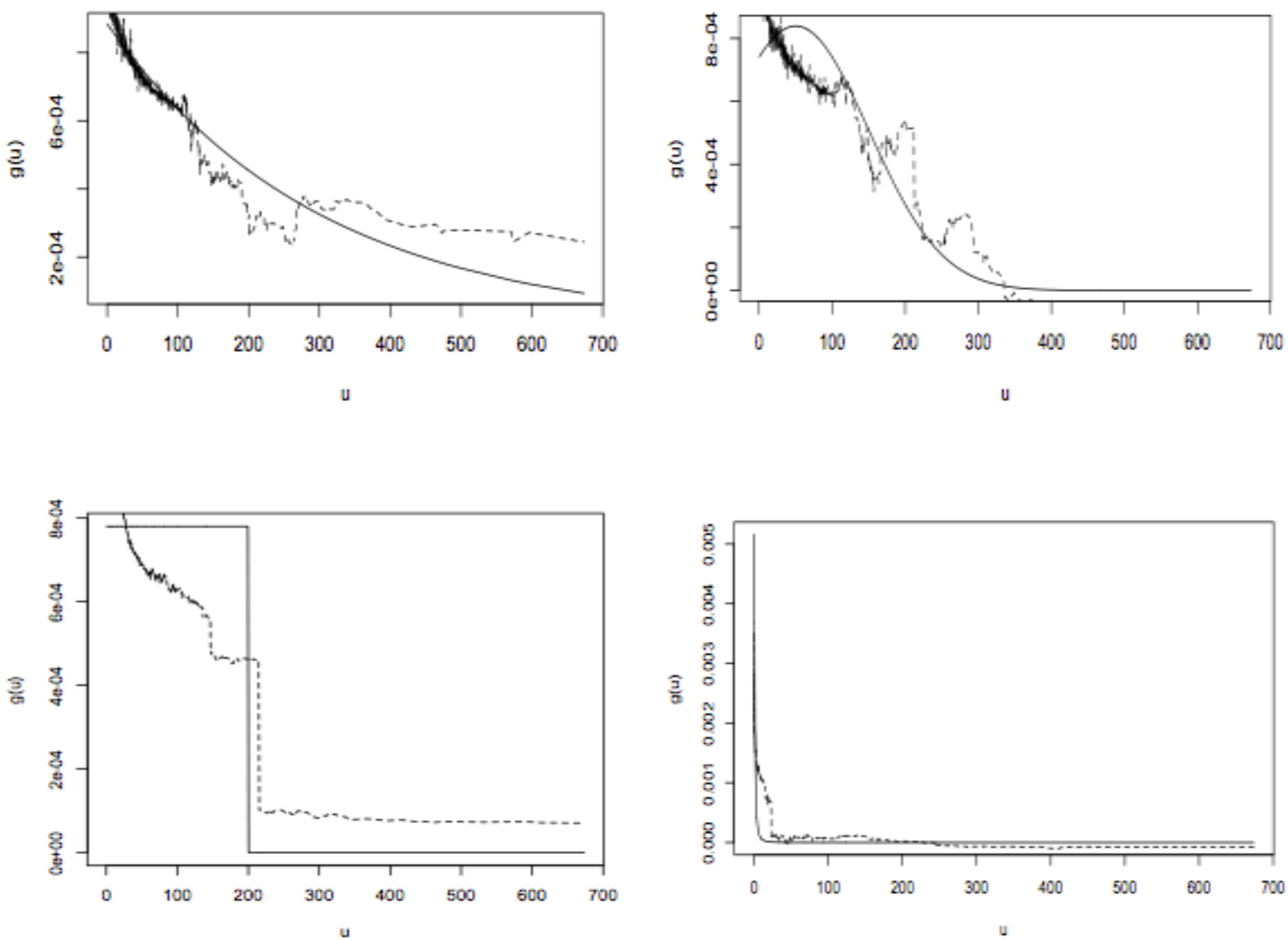
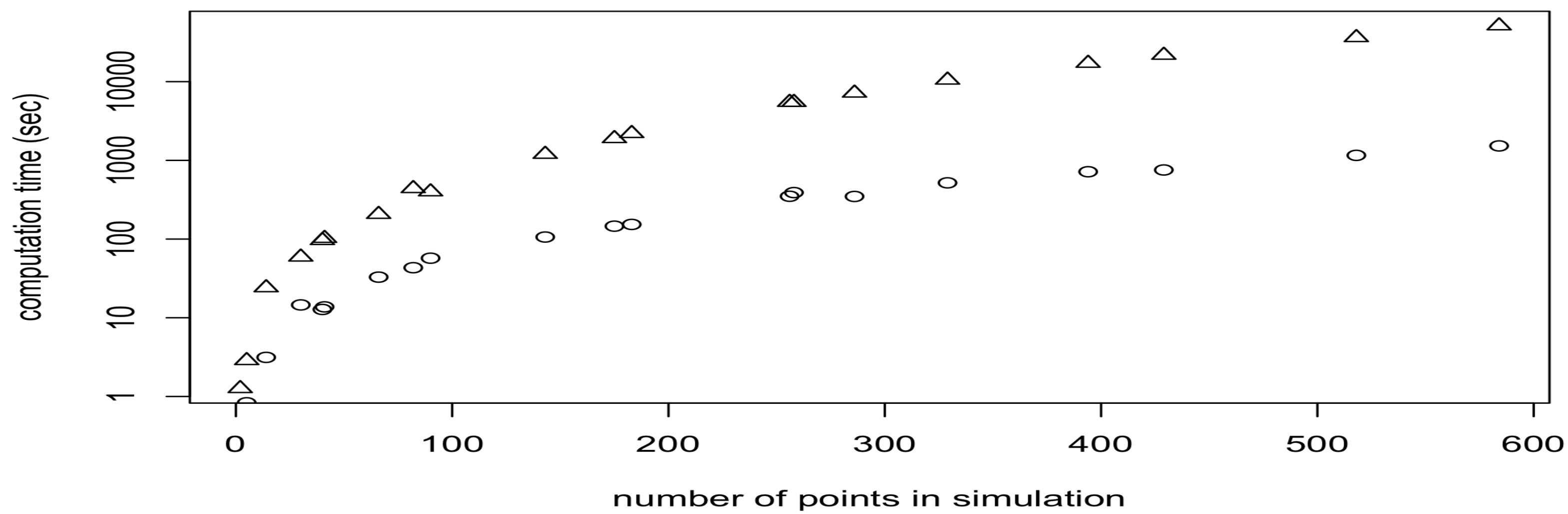
**Computation time and performance comparison.**

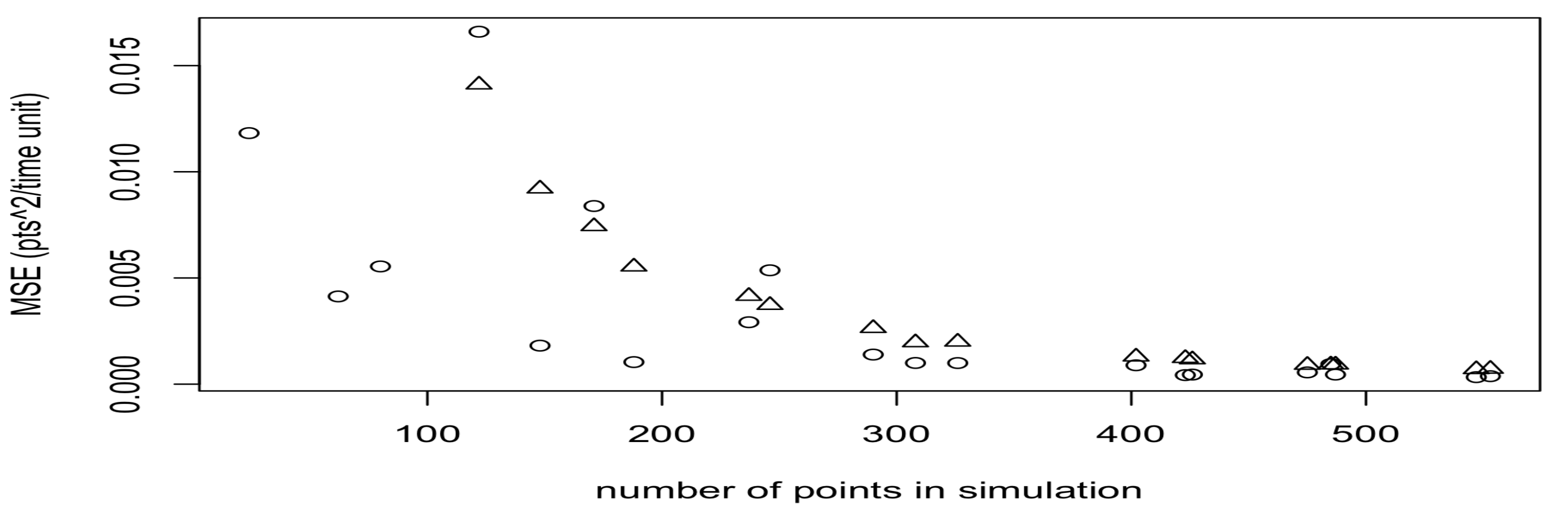
Test of concept. Examples of exponential, truncated normal, uniform, and Pareto $g$.



**Computation time and performance comparison.**

Triangles = Marsan and Lengliné (2008) method. Circles = analytic method.



The idea is to

let p=n,

let the derivatives of the log-likelihood be zero,

solve for $1/\lambda_i$ and therefore get $\lambda_i$,

and solve for $\beta$.

a. One can have major computation time savings from this method.

For datasets of only 100-300 points the savings are negligible.

However, for 5,000 points, the Marsan and Lengliné (2008) algorithm with 100 iterations takes about 7 hours, whereas the analytic method takes 1.3 min.

This speed facilitates computations like simulation based confidence intervals.

b. How far can this go?

It extends very readily to space-time-magnitude and estimation of $\mu$ .

Would this work for other types of models too? What are the limits on this method?

c. What about when A is singular? More work is needed.