Statistics 222, Spatial Statistics.

Outline for the day:

1. Variance of compound Poisson.
2. Gibbs process.
3. Strauss process.
4. Matern process.
5. Hawkes process.
6. ETAS.
7. Likelihood.

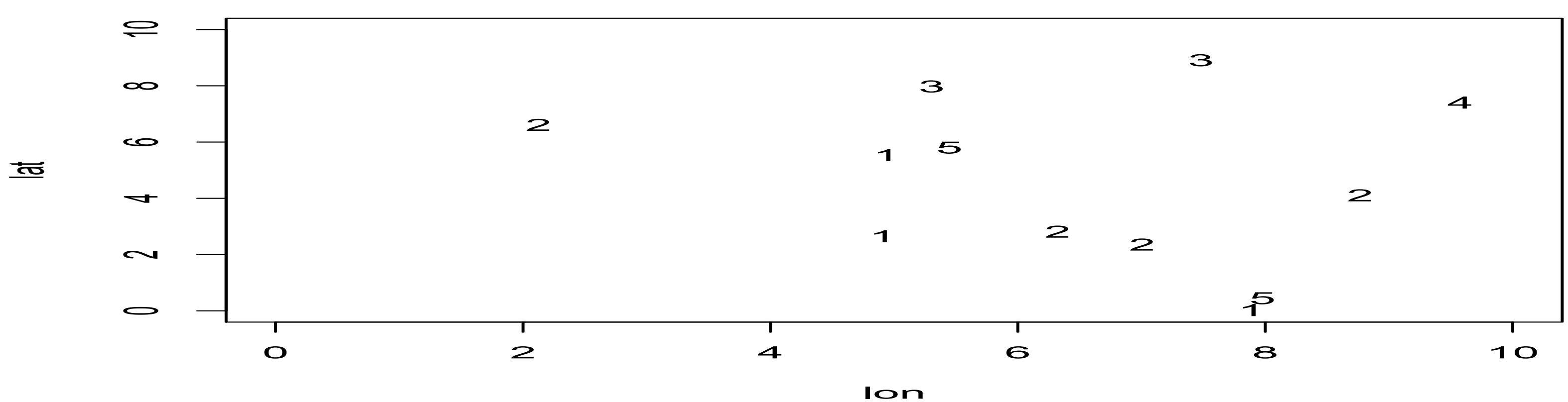
1. Variance of the compound Poisson processes, from last time.

Suppose N is not simple, and instead, it is generated as follows. You first generate a stationary Poisson process M with intensity c, and then for each point τ_i of M, N will have some non-negative number Z_i of points right at τ_i, where Z_i are all iid and independent of M. Then N is a compound Poisson process.

For a compound Poisson process, again the variance $\geq$ the mean.

$EN(B) = c|B|E(Z)$, and

$V(N(B)) = c|B|V(Z) + c|B|(E(Z))^2 = c|B|E(Z^2) \geq EN(B)$, because, for a non-negative integer-valued random variable Z, $E(Z^2) \geq E(Z)$ with equality iff. Z can only be 0 or 1.



1. Variance of the compound Poisson processes, from last time.

Fix B. Let M denote M(B). For a compound Poisson process,

$$
\begin{aligned}
EN(B) &= \sum E(N(B)|m)\, f(m) \text{, where the sum is from } m = 0, 1, 2, \ldots, \\
&= \sum E(Z1 + Z2 + \ldots + Zm)\, f(m) \\
&= \sum (m\, E(Z))\, f(m) \\
&= E(Z) \sum m\, f(m) \\
&= E(Z)\, E(M) = c|B|\, E(Z).
\end{aligned}
$$

$$
\begin{aligned}
E(N(B)^2) &= \sum E(N(B)^2|m)\, f(m) \\
&= \sum E(Z1 + Z2 + \ldots + Zm)^2\, f(m) \\
&= \sum (mE(Z^2) + (m^2\text{-}m)\, E(Z)^2) f(m) \\
&= E(Z^2) \sum mf(m) - E(Z)^2 \sum m\, f(m) + E(Z)^2 \sum m^2\, f(m) \\
&= E(Z^2)\, E(M) - E(Z)^2\, E(M) + E(Z)^2\, E(M^2) \\
&= V(Z)E(M) + E(Z)^2\, E(M^2).
\end{aligned}
$$

$$
\begin{aligned}
\text{So } V(N(B)) &= E(N(B)^2) - (E(N(B)))^2 \\
&= V(Z)E(M) + E(Z)^2\, E(M^2) - E(M)^2\, E(Z)^2 \\
&= V(Z)\, E(M) + E(Z)^2\, (E(M^2) - E(M)^2\,) \\
&= V(Z)\, E(M) + E(Z)^2\, V(M).
\end{aligned}
$$

M is Poisson, so $E(M) = V(M) = c|B|$, so

$V(N(B)) = c|B|\, (V(Z) + E(Z)^2) = c|B|\, E(Z^2) \geq EN(B)$, since $E(Z^2) \geq E(Z)$.

2. Gibbs process.

For any finite collection $(\tau_1, \tau_2, ..., \tau_n)$ of points in space-time, if the joint density is
$C(\theta) \exp[-\theta \{\sum_i \psi_1(\tau_i) + \sum_{i,j} \psi_2(\tau_i,\tau_j)\}]$,
then N is a Gibbs process.

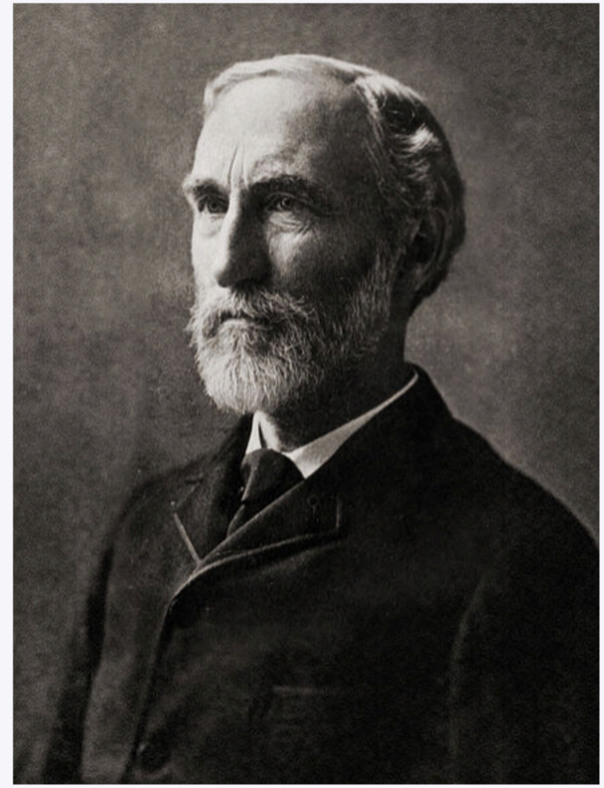
Often $\psi_2(x_i , x_j)$ can be written $\psi(r)$, where $r = |x-y|$.

Some special cases are important.

a. When $\psi(r) = 0$, there are no interactions and the process is an inhomogeneous Poisson process with intensity $\psi_1(x)$.

b. $\psi(r) = -\log[1-e^{-(r/\sigma)^2}]$ defines a *soft-core* model. Weak repulsion.

Josiah Willard Gibbs



Josiah Willard Gibbs

Born	February 11, 1839 New Haven, Connecticut, U.S.
Died	April 28, 1903 (aged 64) New Haven, Connecticut, U.S.

2. Gibbs process, continued.

$\psi_2(r)$ is called the *interaction potential*.

c. $\psi(r) = \infty$ for $r \leq \sigma$

$= 0$ for $r > \sigma$

defines a *hard-core* process.

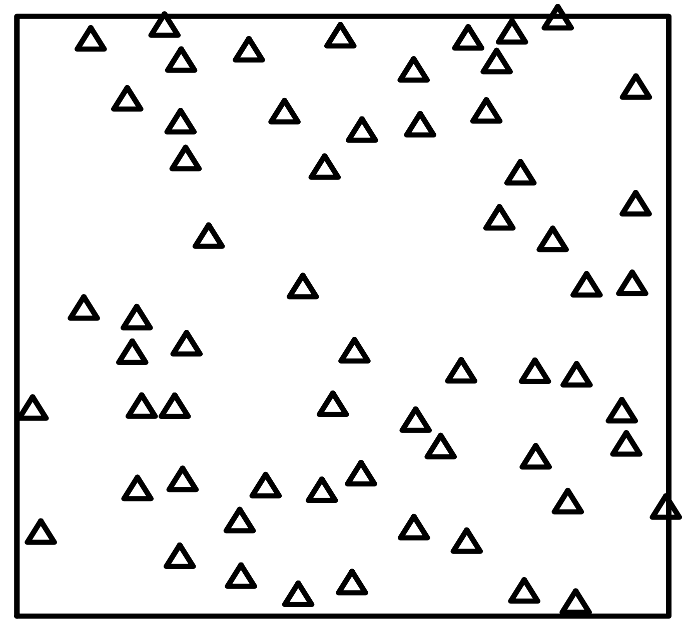
d. $\psi(r) = (\sigma/r)^n$ is an intermediate choice between the soft-core and hard-core models.

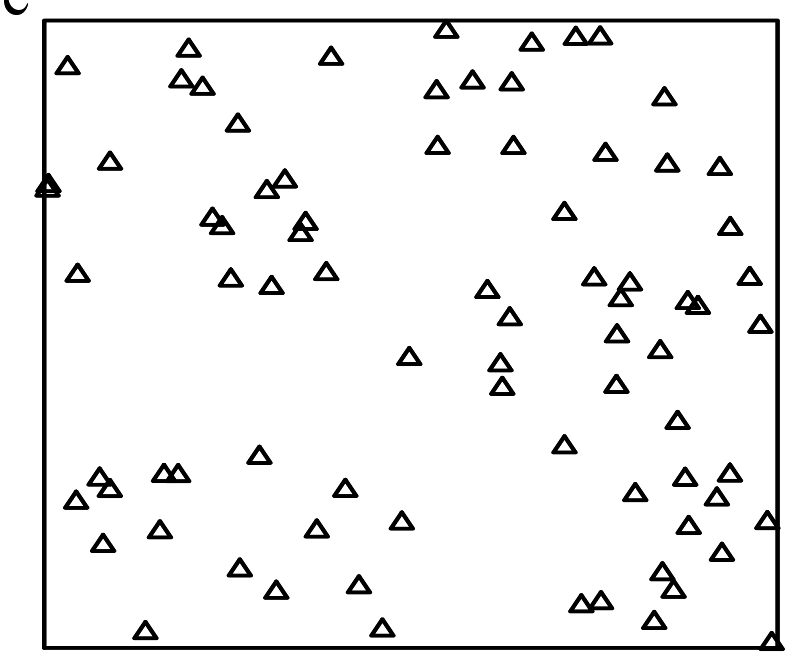
3. Strauss process.

$\psi_1(x) = \alpha$, and

$\psi_2(r) = \beta$, for $r \leq R$,

$\psi_2(r) = 0$, for $r > R$.





4. Matern process.

The Matern(I) process is generated as follows.

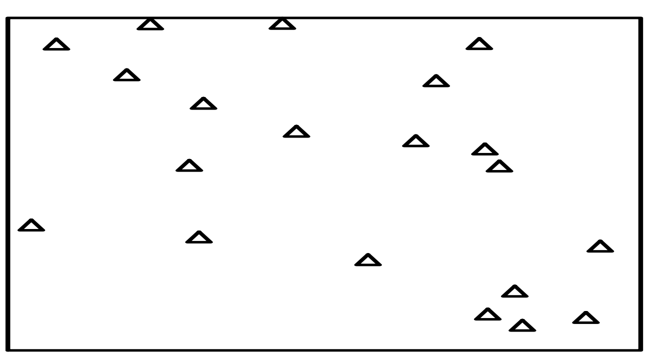
a) Generate M according to a stationary Poisson process.

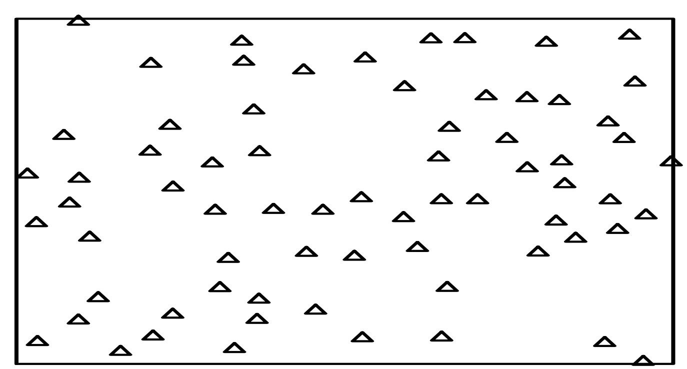
b) Let N be all points of M that are not within some fixed distance r of any other point of M.

The Matern(II) process is generated a bit differently.

a) Generate points τ_1, τ_2, ... according to a stationary Poisson process.

b) For i = 1, 2, ..., keep point i if there is no *previous* kept point τ_j with $|\tau_i - \tau_j| \leq r$.





5. Hawkes process.

A Hawkes process or *self-exciting* process has conditional intensity

$$\lambda(t,x,y) = \mu(x,y) + \kappa \int_{t'<t} g(t-t',x-x',y-y')\, dN(t',x',y')$$

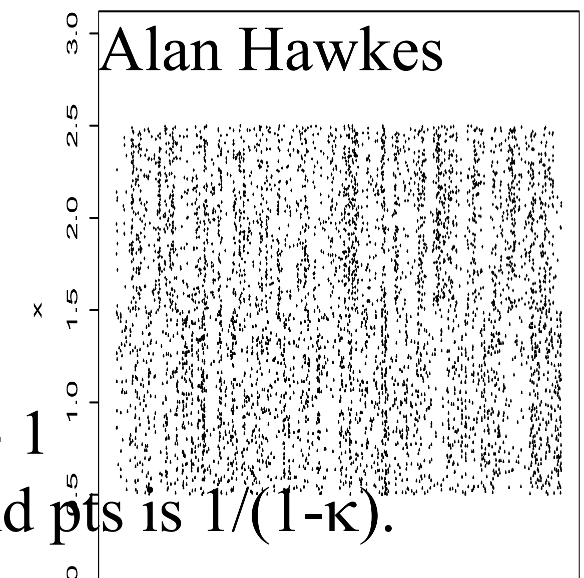
$$= \mu(x,y) + \kappa \sum_{\{t',x',y':\ t' < t\}} g(t-t',x-x',y-y').$$

g is called the *triggering function* or *triggering density* and κ is the *productivity*.

If g is a density function, then κ is the expected number of points triggered directly by each point.

Each background point, associated with $\mu(x,y)$, is expected to generate $\kappa + \kappa^2 + \kappa^3 + \ldots = 1/(1-\kappa) - 1$ triggered points, so the exp. fraction of background pts is $1/(1-\kappa)$.

Alan Hawkes



6. ETAS process.

Yosihiko Ogata

An *Epidemic-Type Aftershock Sequence (ETAS)* process is a marked version of the Hawkes process, where points have different productivities depending on their magnitudes. Ogata (1988, 1998) introduced

$$\lambda(t,x,y)=\mu(x,y) + \sum_{\{t',x',y':\ t' < t\}} g(t-t',x-x',y-y')h(m'),$$

where $\mu(x,y)$ is estimated by smoothing observed large earthquakes,

$h(m) = \kappa\, e^{\alpha(m-m0)}$,

where m0 is the catalog cutoff magnitude,

and $g(t,x,y) = g_1(t)\, g_2(r^2)$, where $r^2 = ||(x,y)||^2$,

and g_1 and g_2 are power-law or *Pareto* densities,

$g_1(t) = (p-1)\, c^{p-1}\, (t+c)^{-p}$. $g_2(r^2) = (q-1)\, d^{q-1}\, (r^2+d)^{-q}$.

An alternative is where g2 is exponential or sum of exponentials.

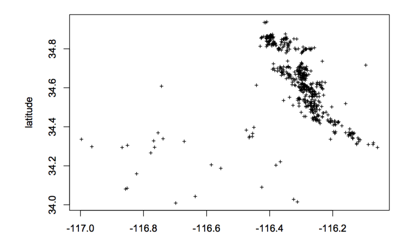


Figure 3: Recorded epicenters of Hector Mine $M \geq 3.0$ earthquakes from 10/16/1999 to 12/23/2000, from SCSN.

7. Likelihood.

For iid draws $t_1, t_2, \ldots, t_n$, from some density $f(\theta)$, the likelihood is simply
$L(\theta) = f(t_1 ; \theta) \times f(t_2 ; \theta) \times \ldots \times f(t_n ; \theta)$
$= \prod f(t_i ; \theta)$.

This is the probability density of observing $\{t_1, t_2, \ldots, t_n\}$, as a function of the parameter θ.

For a stationary Poisson process with intensity $\lambda(\theta)$, on $[0,T]$, the likelihood of observing the points $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is simply
$\lambda(\tau_1) \times \lambda(\tau_2) \times \ldots \times \lambda(\tau_n) \times$
$\exp\{-A(\tau_1)\} \times \exp\{-(A(\tau_2)-A(\tau_1))\} \times \ldots \times \exp\{-(A(T)-A(\tau_n))\}$,
$= \prod \lambda(\tau_i) \exp\{-A(T)\}$,
where $A(t) = \int_0^t \lambda(t)dt$.
$P\{k \text{ points in } (\tau_1, \tau_2)\}$ is $\exp(-B)\, B^k/k! = \exp(-B)$ for $k = 0$,
where $B = \int_{\tau 1}^{\tau 2} \lambda(t)dt$.

7. Likelihood, continued.

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$\lambda(\tau_1) \times \lambda(\tau_2) \times \ldots \times \lambda(\tau_n) \times$

$$\exp\{-A(\tau_1)\} \times \exp\{-(A(\tau_2)-A(\tau_1))\} \times \ldots \times \exp\{-(A(T)-A(\tau_n))\},$$

$= \prod \lambda(\tau_i) \exp\{-A(T)\}$,
where $A(t) = \int_0^t \lambda(t)dt$.
$P\{k \text{ points in } (\tau_1, \tau_2)\}$ is $\exp(-B)\, B^k/k! = \exp(-B)$ for $k = 0$,
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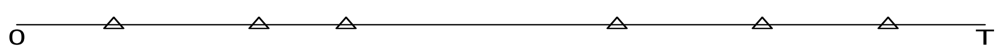

So the log likelihood is $\sum \log(\lambda(\tau_i))$ $-A(T)$.
In the spatial-temporal case, the log likelihood is simply
$\sum \log(\lambda(\tau_i)) - \int \lambda(t,x,y)\, dt\, dx\, dy$.

8. Questions.

The difference between ETAS and a Hawkes process is ...

a) an ETAS process is more strongly clustered.
b) the points of an ETAS process all occur at different locations.
c) the points of an ETAS process all have different productivity.
d) the points of an ETAS process all have different triggering functions.

8. Questions.

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a) an ETAS process is more strongly clustered.
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d) the points of an ETAS process all have different triggering functions.

Which of the following can possibly have two points within distance .01 of each other?

a) a hardcore process with $\sigma = .01$.
b) a Strauss process with $R = .01$.
c) a Matern I process with $r = .01$.
d) a Matern II process with $r = .01$.

8. Questions.

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a) a hardcore process with $\sigma = .01$.
b) a Strauss process with R = .01.
c) a Matern I process with $r = .01$.
d) a Matern II process with $r = .01$.

9. Code.

```
install.packages("spatstat")
library(spatstat)

## STRAUSS process
z = rStrauss(100,0.7,0.05)
plot(z, pch=2,cex=.5)

## HARDCORE process
z = rHardcore(100,0.05)
plot(z, pch=2,cex=.5)

## MATERN(I).
z = rMaternI(20,.05)
plot(z, pch=2,cex=.5)
```

9. Code.

```
## MATERN(II)
z = rMaternII(100,.05)
plot(z,pch=2,cex=.5)

## HAWKES.
install.packages("hawkes")
library(hawkes)
lambda0 = c(0.2,0.2)
alpha = matrix(c(0.5,0,0,0.5),byrow=TRUE,nrow=2)
beta = c(0.7,0.7)
horizon = 3600
h = simulateHawkes(lambda0,alpha,beta,horizon)
plot(c(0,3600),c(0,3),type="n",xlab="t",ylab="x")
points(h[[1]],.5+runif(length(h[[1]])),pch=2,cex=.1)
points(h[[2]],1.5+runif(length(h[[2]])),pch=3,cex=.1)
```