Statistics 222, Spatial Statistics.

Outline for the day:

1. Poisson process, continued.

2. Mixed Poisson process.

3. Compound Poisson process

4. Poisson cluster process.

5. Cox process.

6. Gibbs processes.

7. Matern processes.

8. Examples and code.

1. Poisson processes.

Last week we discussed Poisson processes.

If N is a simple point process with conditional intensity λ, where λ does not depend on what points have occurred previously, then N is a *Poisson process*.

For such a process, for any set B,
N(B) has a Poisson distribution.
$P(N(B) = k) = e^{-A} A^k / k!$,
for $k = 0, 1, 2, ...,$
where $A = \int_B \lambda(t,x,y)\, dtdxdy$,
and with the convention $0! = 1$.
The mean of N(B) is A and the variance is also A. $E(N(B)^2) = A^2 + A$.

Siméon Denis Poisson (1781–1840)

We will now discuss a few extensions of Poisson processes.

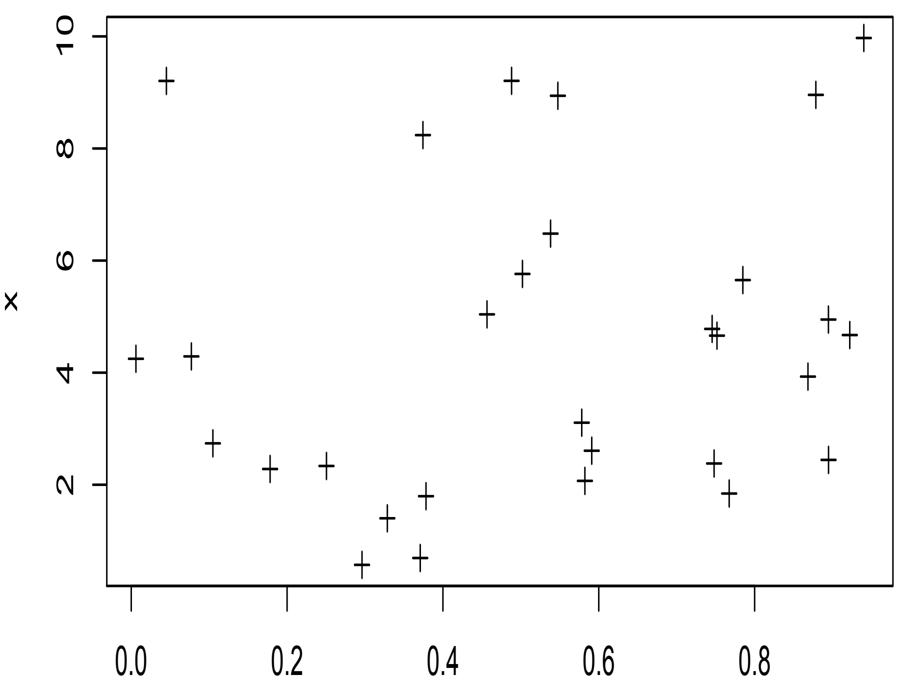
Poisson processes, continued.

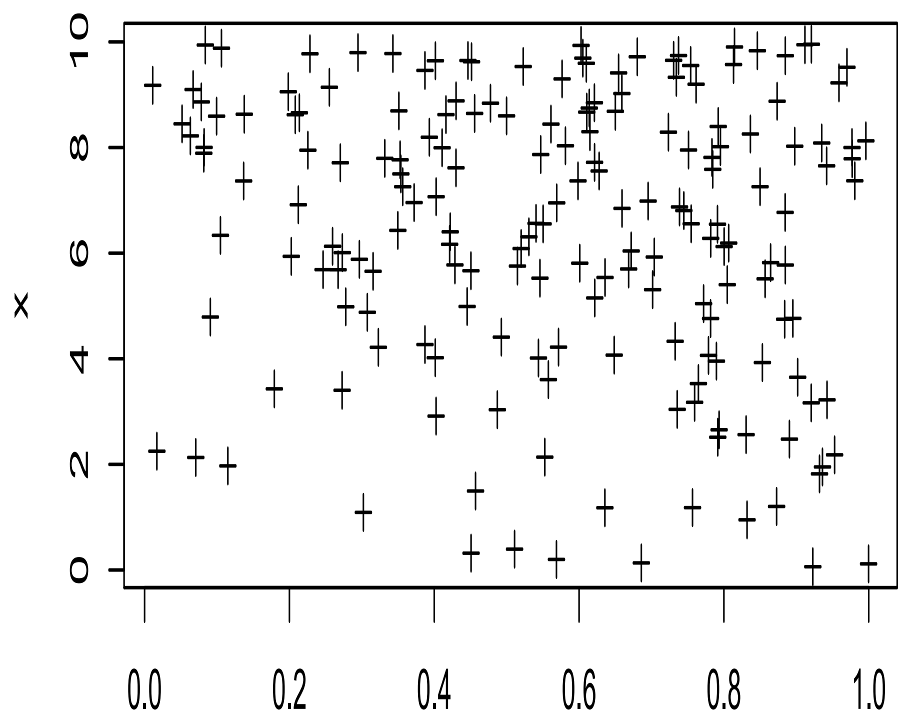
On the left is a stat. Poisson process with $\lambda(t,x) = 2.5$ on $[0,1] \times [0,10]$,

and on the right is a Poisson process with $\lambda(t,x) = 1.5 + 10t + 2x$.

The key thing about Poisson processes is their complete independence.

For a Poisson process N, $N(B_1)$ and $N(B_2)$ are independent for any disjoint sets B_1 and B_2.

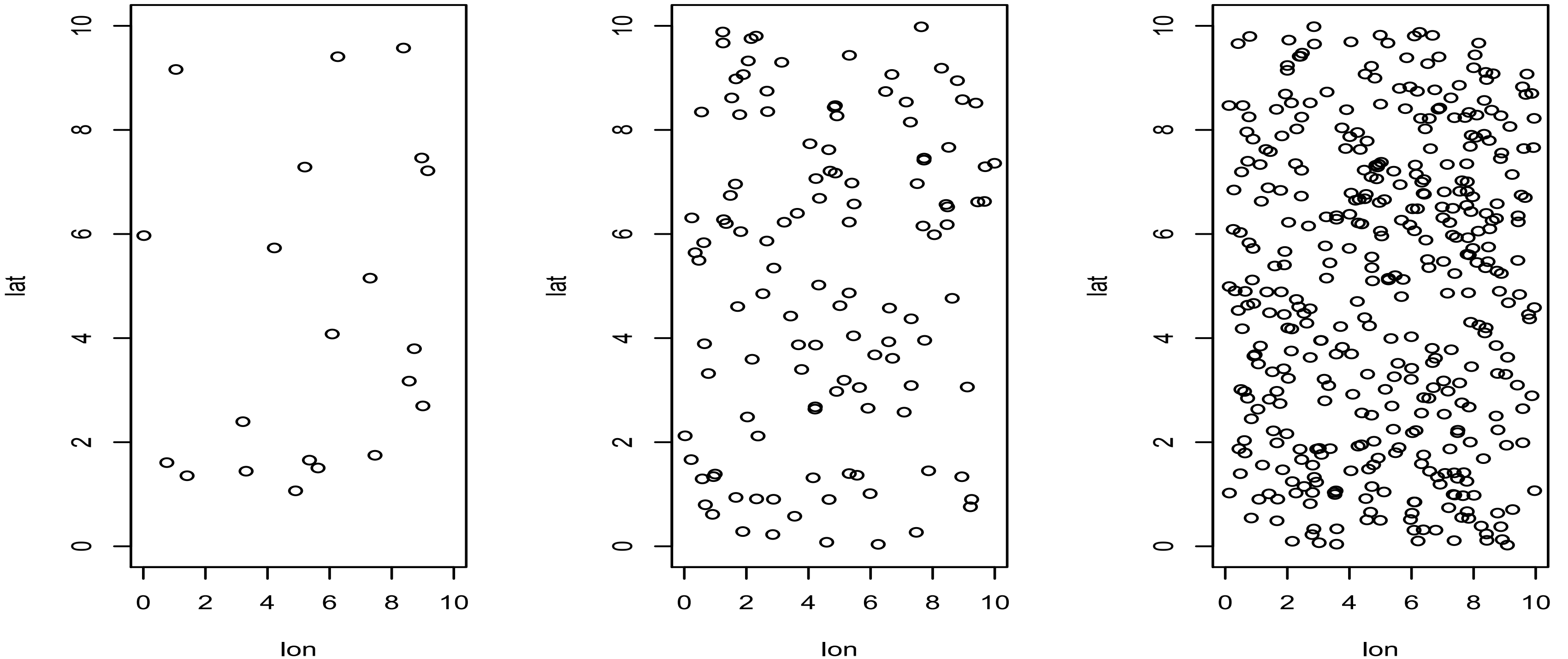




2. Mixed Poisson processes.

Suppose $\lambda(t,x,y) = c$, where c is a random variable. For example, c might be Poisson or exponential, or half normal, or something constrained to be positive. Then conditional on c, $N(B)$ is Poisson distributed. Then N is a *mixed Poisson process*.

$E(N(B) \mid c) = V(N(B)|c) = c|B|$, but unconditionally, $N(B)$ is not Poisson distributed now.



2. Mixed Poisson processes.

Suppose $\lambda(t,x,y) = c$, where c is a random variable. For example, c might be Poisson or exponential, or half normal, or something constrained to be positive. Then conditional on c, $N(B)$ is Poisson distributed. Then N is a *mixed Poisson process*.

$E(N(B) \mid c) = V(N(B)|c) = c|B|$, but unconditionally, $N(B)$ is not Poisson distributed now. If we imagine simulating the process repeatedly, each time with a different draw of c, then the distribution of $N(B)$ will not be Poisson. $N(B)$ will typically be overdispersed relative to the Poisson process, i.e. will have higher variance.

$E(N(B)) = \int E(N(B)|c)\ f(c)dc = \int c|B|\ f(c)dc = |B|E(c).$

$E(N(B)^2) = \int E(N(B)|c)^2\ f(c)\ dc = \int [c^2|B|^2 + c|B|]\ f(c)\ dc$

$= |B|^2\ E(c^2) + |B|E(c),$

so $V(N(B)) = |B|^2\ E(c^2) + |B|E(c) - |B|^2\ [E(c)]^2 = E(N(B)) + |B|^2\ V(c).$

So, $V(N(B)) \geq E(N(B)).$

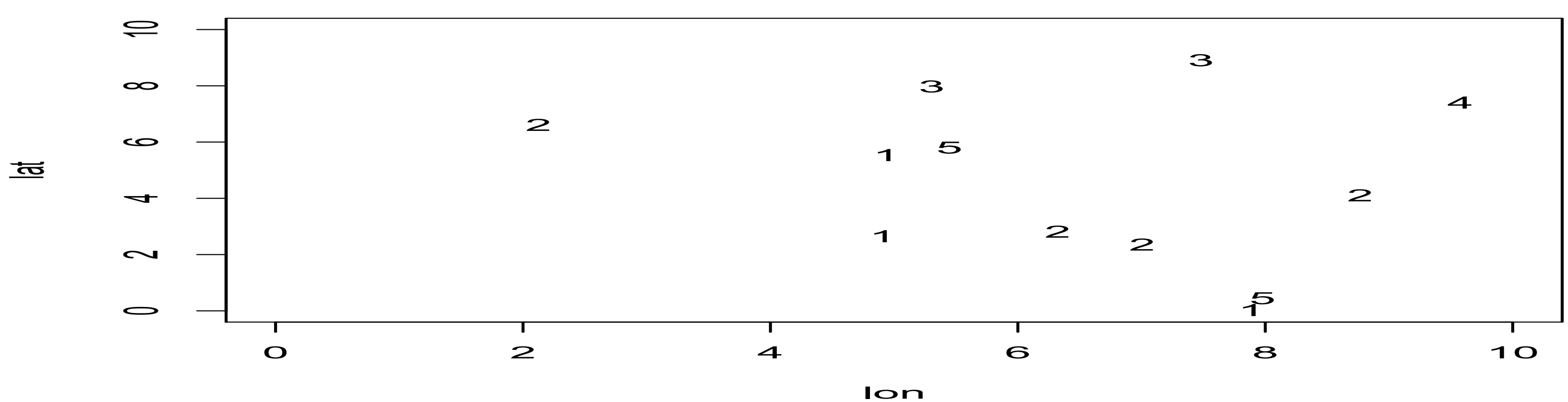
3. Compound Poisson process.

Suppose N is not simple, and instead, it is generated as follows. You first generate a stationary Poisson process M with intensity c, and then for each point τ_i of M, N will have some non-negative number Z_i of points right at τ_i, where Z_i are all iid and independent of M. Then N is a compound Poisson process.

For a compound Poisson process, again the variance $\geq$ the mean.

$EN(B) = c|B|E(Z)$, and

$V(N(B)) = c|B|V(Z) + c|B|(E(Z))^2 = c|B|E(Z^2) \geq EN(B)$, because, for a non-negative integer-valued random variable Z, $E(Z^2) \geq E(Z)$ with equality iff. Z can only be 0 or 1.



3. Variance of the compound Poisson process.

Fix B. Let M denote M(B). For a compound Poisson process,

$EN(B) = c|B|\ E(Z)$.

$E(N(B)^2) = V(Z)\ E(M) + (E(Z))^2\ E(M^2)$.

So $V(N(B)) = V(Z)\ E(M) + E(Z)^2\ V(M)$.

M is Poisson, so $E(M) = V(M) = c|B|$, so

$V(N(B)) = c|B|\ (V(Z) + E(Z)^2\ ) = c|B|\ E(Z^2) \geq\ EN(B)$,

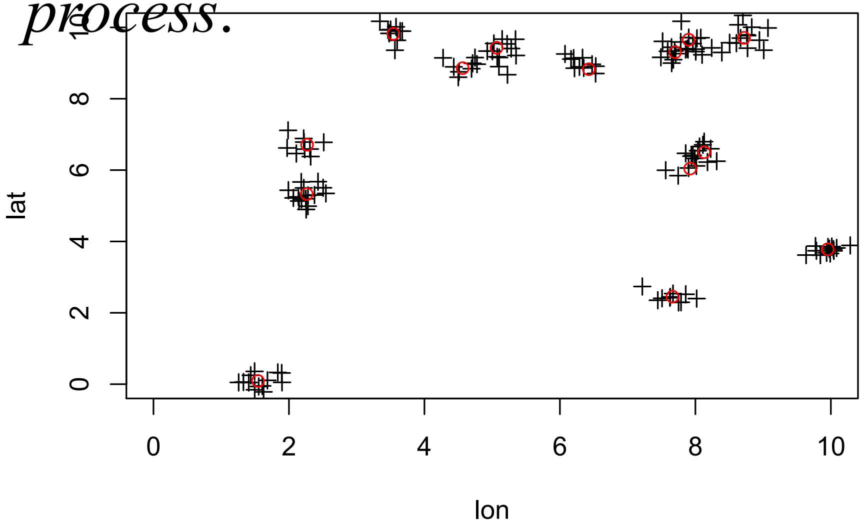
since $E(Z^2)\ \geq E(Z)$ because Z is nonnegative integer valued.

4. Poisson cluster processes.

Another extension of the Poisson process is the Poisson cluster process. Imagine first generating *parent* points *M* according to a Poisson process. Then for each parent point τ_i, you generate some random number Z_i of offspring points, and these offspring points are scattered spatially and temporally, independently of each other, with some distribution centered at τ_i. Let *N* be the collection of just the offspring, not the parents. *N* is called *Poisson cluster process*.

Usually *M* is assumed *stationary* Poisson.

In the particular case where the Z_i are iid Poisson random variables independent of *M*, the process is called a *Neyman-Scott cluster process*.

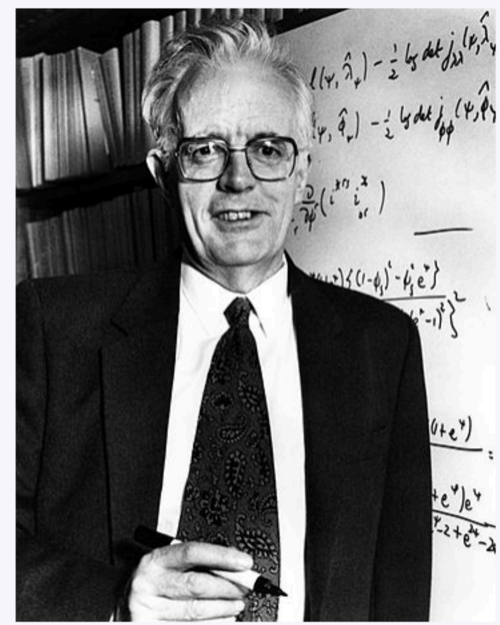


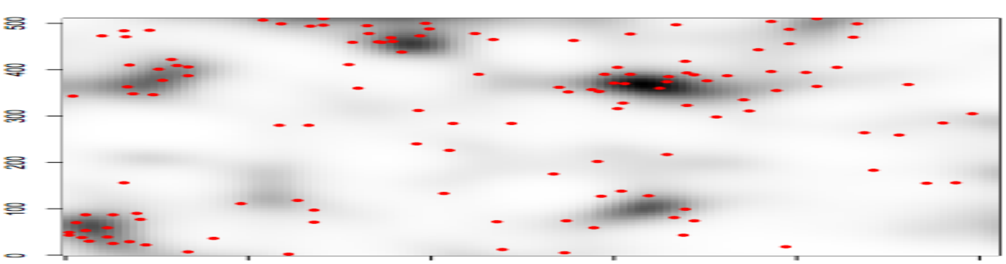
5. Cox process.

Suppose you somehow generate a stochastic process λ(t,x,y) such that λ(t,x,y) ≥ 0 for all t, x, and y. Then you let *N* be a Poisson process with intensity λ(t,x,y). So λ(t,x,y) can be random, but conditional on λ, *N* is a Poisson process. In this case we say *N* is a *Cox* process or equivalently a *doubly stochastic Poisson process.*

Cox processes arise in practice when modeling events depending on some other random phenomenon. For instance, the points of *N* might be the times and locations of flu epidemics, which might depend on the temperature and this might in turn be modeled as evolving stochastically.

David Cox





6. Gibbs process.

For any finite collection $(\tau_1, \tau_2, ..., \tau_n)$ of points in space-time, if the joint density is $C(\theta) \exp[-\theta \{\sum_i \psi_1(\tau_i) + \sum_{i,j} \psi_2(\tau_i,\tau_j)\}]$, then N is a Gibbs process.

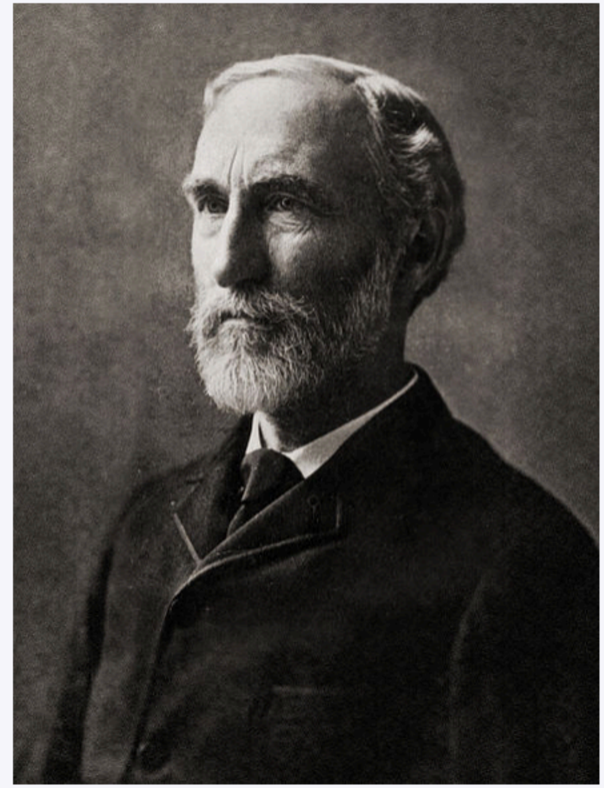
Often $\psi_2(x_i , x_j)$ can be written $\psi(r)$, where $r = |x_i - x_j|$.

Some special cases are important.

a. When $\psi(r) = 0$, there are no interactions and the process is an inhomogeneous Poisson process with intensity $\psi_1(x)$.

b. $\psi(r) = -\log[1-e^{-(r/\sigma)^2}]$ defines a *soft-core* model. Weak repulsion.

Josiah Willard Gibbs



Josiah Willard Gibbs

Born	February 11, 1839 New Haven, Connecticut, U.S.
Died	April 28, 1903 (aged 64) New Haven, Connecticut, U.S.

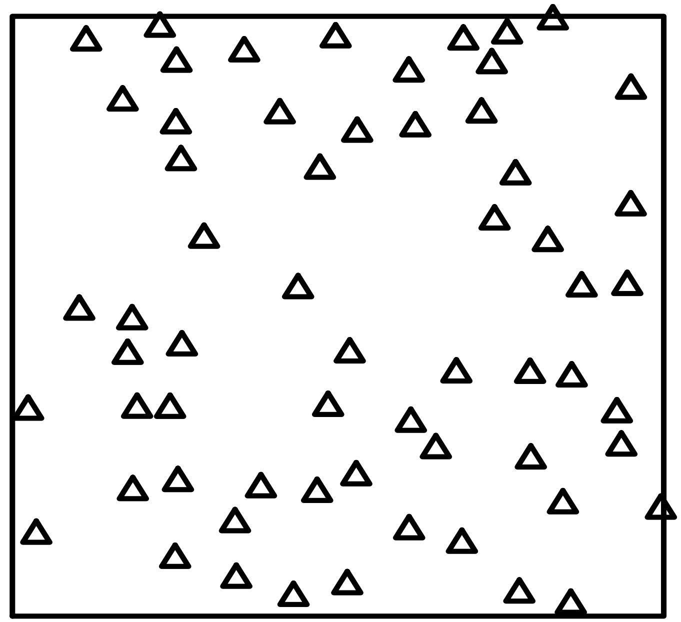
6. Gibbs process, continued.

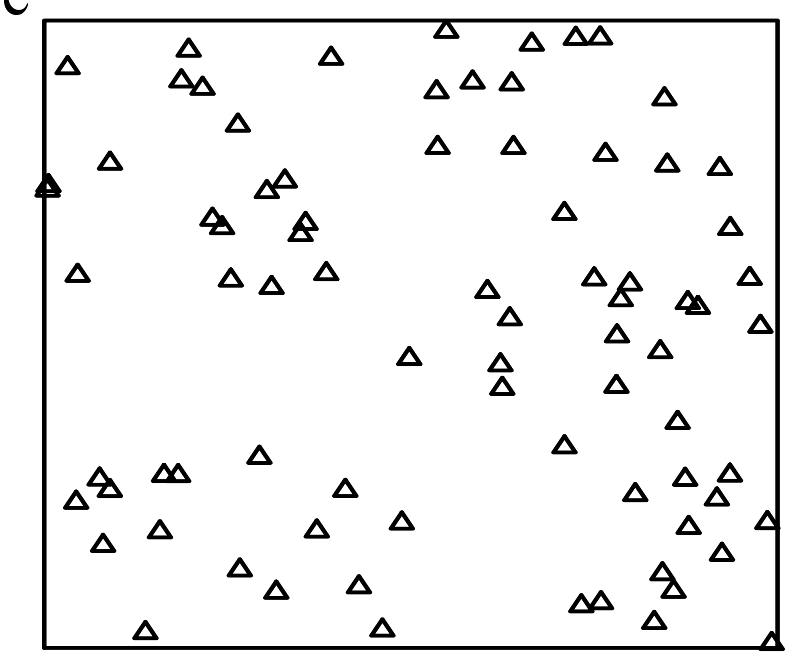
$\psi_2(r)$ is called the *interaction potential*.

c. $\psi(r) = \infty$ for $r \leq \sigma$
$= 0$ for $r > \sigma$
defines a *hard-core* process.

d. $\psi(r) = (\sigma/r)^n$ is an intermediate choice between the soft-core and hard-core models.

e. Strauss process.
$\psi_1(x) = \alpha$, and
$\psi_2(r) = \beta$, for $r \leq R$,
$\psi_2(r) = 0$, for $r > R$.





7. Matern process.

The Matern(I) process is generated as follows.

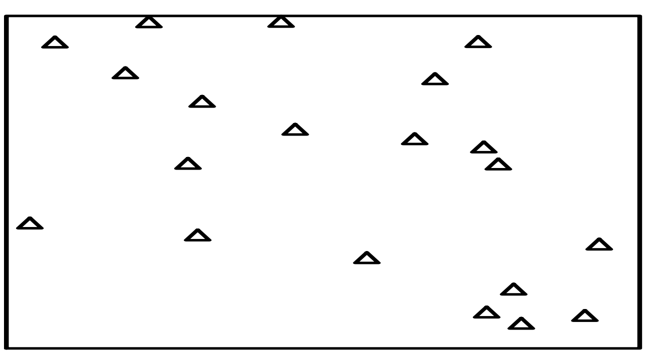
a) Generate M according to a stationary Poisson process.

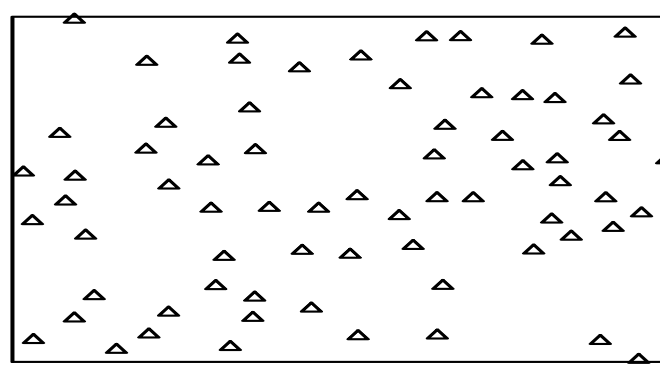
b) Let N be all points of M that are not within some fixed distance r of any other point of M.

The Matern(II) process is generated a bit differently.

a) Generate points $\tau_1, \tau_2, \ldots$ according to a stationary Poisson process.

b) For i = 1, 2, ..., keep point i if there is no *previous* kept point τ_j with $|\tau_i - \tau_j| \leq r$.





Exercises.

1. A mixed Poisson process is a Cox process where

a. $\lambda = E(\lambda)$ in every realization.

b. $\lambda(t,x,y) = \lambda(t',x',y')$, for any locations (t,x,y) and (t',x',y').

c. The cluster sizes are Poisson distributed with mean λ.

d. $\lambda = 1$.

Exercises.

1. A mixed Poisson process is a Cox process where

a. $\lambda = E(\lambda)$ in every realization.

b. $\lambda(t,x,y) = \lambda(t',x',y')$, for any locations (t,x,y) and (t',x',y').

c. The cluster sizes are Poisson distributed with mean λ.

d. $\lambda = 1$.

a. means λ is a constant, so *N* is a stationary Poisson process.
d. Also defines a stationary Poisson process, with rate 1.

Code from Day 2.

```
## nonsimple point process
n = 20
x = runif(n)
y = runif(n)
plot(x,y,xlab="t",ylab="lat",pch=2)
points(x[20],y[20],pch=3)

## nonsimple ground process
plot(x,y,xlab="t",ylab="lat",pch=2)
points(x[20],y[20]+.05,pch=3)

## nonorderly process
plot(c(0,1),c(0,1),type="n",xlab="t",ylab="lat")
n = 100
for(i in 1:n) points(1/i,runif(1),pch=3,cex=.5)
```

Code from Day 2.

```
## points at (i,i) with prob. 1/i.
plot(c(0,100),c(0,100),type="n",xlab="t",ylab="lat")
for(i in 1:100) if(runif(1) < 1/i) points(i,i,pch=3)

## stationary Poisson process with intensity 2.5 on B=[0,1]x[0,10].
n = rpois(1,2.5*1*10)
t = runif(n)
x = runif(n)*10
plot(t,x,pch=3)
```

Code from Day 2.

```
## nonstationary Poisson process with intensity 1.5+10t+2x on B.
n = rpois(1,15+50+100)
n1 = 0
t = c()
x = c()
while(n1<n){
t2 = runif(1) ## candidate point
x2 = runif(1)*10
if(runif(1) < (1.5+10*t2+2*x2)/(1.5+10+20)){ ## keep it
  t = c(t,t2)
  x = c(x,x2)
  n1 = n1 + 1
  cat(n1," ")
}
}
plot(t,x,pch=3)
```

Code from Today.

```
## mixed Poisson process
par(mfrow=c(1,3))
m = rexp(1,rate=.5)
n1 = rpois(1,m*10*10)
x1 = runif(n1)*10
y1 = runif(n1)*10
plot(c(0,10),c(0,10),xlab="lon",ylab="lat",type="n")
points(x1,y1)
## I ran the previous 5 lines 3 times.
```

Code.

```
## compound Poisson.
par(mfrow=c(1,1))
n1 = rpois(1,.12*10*10)
x1 = runif(n1)*10
y1 = runif(n1)*10
a = as.character(rpois(n1,3))
plot(c(0,10),c(0,10),xlab="lon",ylab="lat",type="n")
text(x1,y1,a)
```

Code.

```
## Neyman-Scott.
n1 = rpois(1,.12*10*10)
x1 = runif(n1)*10
y1 = runif(n1)*10
x2 = c()
y2 = c()
## parents are (x1,y1).
for(i in 1:n1){
c = rpois(1,8) ## number of offspring
if(c>0) for(j in 1:c){
x2 = c(x2,rnorm(1,sd=.2)+x1[i])
y2 = c(y2,rnorm(1,sd=.2)+y1[i])
}}
plot(c(0,10),c(0,10),xlab="lon",ylab="lat",type="n")
points(x2,y2,pch=3)
points(x1,y1,col="red")
```

1. Variance of the compound Poisson processes, from last time.

Fix B. Let M denote M(B). For a compound Poisson process,

$$
\begin{aligned}
EN(B) &= \sum E(N(B)|m)\, f(m) \text{, where the sum is from } m = 0, 1, 2, \ldots, \\
&= \sum E(Z1 + Z2 + \ldots + Zm)\, f(m) \\
&= \sum (m\, E(Z))\, f(m) \\
&= E(Z) \sum m\, f(m) \\
&= E(Z)\, E(M) = c|B|\, E(Z).
\end{aligned}
$$

$$
\begin{aligned}
E(N(B)^2) &= \sum E(N(B)^2|m)\, f(m) \\
&= \sum E(Z1 + Z2 + \ldots + Zm)^2\, f(m) \\
&= \sum (mE(Z^2) + (m^2\text{-}m)\, E(Z)^2) f(m) \\
&= E(Z^2)\sum mf(m) - E(Z)^2 \sum m\, f(m) + E(Z)^2 \sum m^2\, f(m) \\
&= E(Z^2)\, E(M) - E(Z)^2\, E(M) + E(Z)^2\, E(M^2) \\
&= V(Z)E(M) + E(Z)^2\, E(M^2).
\end{aligned}
$$

$$
\begin{aligned}
\text{So } V(N(B)) &= E(N(B)^2) - (E(N(B)))^2 \\
&= V(Z)E(M) + E(Z)^2\, E(M^2) - E(M)^2\, E(Z)^2 \\
&= V(Z)\, E(M) + E(Z)^2\, (E(M^2) - E(M)^2\,) \\
&= V(Z)\, E(M) + E(Z)^2\, V(M).
\end{aligned}
$$

M is Poisson, so $E(M) = V(M) = c|B|$, so

$V(N(B)) = c|B|\, (V(Z) + E(Z)^2) = c|B|\, E(Z^2) \geq EN(B)$, since $E(Z^2) \geq E(Z)$.