Stat 415 midterm is Thu May 7, 6pm to 730pm.

You may need to know that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for x between -1 and 1.
And that the roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $i = \sqrt{-1}$.
And that $\| a + bi \| = \sqrt{a^2 + b^2}$.
You need to also know basics of Variance and Covariance.

You can use any notes or books. Either pen or pencil is fine.

You cannot use any electronics, including your computer, calculator, phone, etc.

EXAMPLE 1.

AR(2): $X_t = .3 X_{t-1} + .1 X_{t-2} + W_t$. What is ϕ?

We can write this, or any AR(p), as $\phi(B) X_t = Wt$. (ϕ=phi)

What is $\phi(B)$?

$\phi(B) = 1 - .3B - .1B^2$.

Because $(1 - .3B - .1B^2) X_t = X_t - .3 X_{t-1} - .1 X_{t-2}$
$= W_t$.

Remember, $B(X_t) = X_{t-1}$, for all t.
$B^2(X_t) = B (BX_t) = B(X_{t-1}) = X_{t-2}$.

Imagine considering f the function of z that corresponds to ϕ.

$\phi(B) = 1 - .3B - .1B^2$,

so it is natural to think about the corresponding function being

$f(z) = 1 - .3z - .1z^2$.

ϕ is a function of B. f is a function of z.
The inverse of the function ϕ, when it is invertible, is called ψ. (ψ=psi).

Often you can match coefficients to find ψ.

EXAMPLE 2.

$X_t - X_{t-1} = W_t - 1/2\ W_{t-1} - 1/2\ W_{t-2}$. Identify the ARMA(p, q) model, and say if it is causal or invertible.

The AR polynomial is $\phi(z) = 1 - z$, which has root 1.

The MA polynomial is $\theta(z) = 1 - z/2 - z^2/2$, which has roots $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{1/2 \pm \sqrt{1/4+2}}{-1}$

$= -1/2 \pm 1.5$,

or -2 and 1.

Since these polynomials share a common root, they have the common factor $1 - z$.

Factoring these out, the reduced process is $X_t = W_t + 1/2\ W_{t-1}$.

To check that it's the same process,

$X_t - X_{t-1} = W_t + 1/2\ W_{t-1} - W_{t-1} - 1/2\ W_{t-2}$

$= W_t - 1/2\ W_{t-1} - 1/2\ W_{t-2}$.

The reduced representation has AR polynomial $\phi(z) = 1$,

which has no roots, so it is causal,

and MA polynomial $\theta(z) = 1 + z/2$, which has root -2, so it is invertible.

So this is a causal and invertible ARMA(0, 1) process.

EXAMPLE 3.

$X_t - 2X_{t-1} + 2X_{t-2} = W_t - 8/9\ W_{t-1}$. Identify the ARMA(p, q) model, and say if it is causal or invertible.

The AR polynomial is $\phi(z)=1-2z+2z^2$, which has roots $\frac{2 \pm \sqrt{4-8}}{4}$,

or $1/2 \pm 1/2\ i$.

These roots are inside the unit circle because $\sqrt{1/2^2 + 1/2^2} = \sqrt{1/2} < 1$.

So it is not causal.

The MA polynomial is $\theta(z) = 1 - 8z/9$, which has root 9/8.

The only root is outside the unit circle, so it is invertible.

The roots are different, so it is in reduced form.

So this is an ARMA(2, 1) process which is invertible but not causal, because the AR polynomial has a root inside the unit circle.

EXAMPLE 4.

Suppose $X_t = 3X_{t-1} - 2X_{t-2} - W_{t-1} + W_t$, where W is white noise.
a. Is the model in reduced form? If not, what is the reduced model?
b. Is it causal?
c. Is it invertible?

a. The AR polynomial is $\phi(z) = 1 - 3z + 2z^2 = (2z-1)(z-1)$, which has roots z=1/2 and z=1.

Alternatively, the roots are $\frac{-b \pm\sqrt{b^2-4ac}}{2a}$, where a = 2, b = -3, c = 1, so the roots are $[3 +/- \sqrt{(9-8)}]/4 = 3/4 +- 1/4$, or 1/2 and 1.

The MA polynomial is = $\theta(z) = (1-z)$, which has root 1. (1-z) is a common factor. So the model is not in reduced form.
The reduced model has AR polynomial $\phi(z) = -(2z-1) = 1-2z$.
The reduced MA polynomial is $\theta(z) = 1$.

So the reduced model is $X_t = 2X_{t-1} + W_t$.

b. The reduced model has AR polynomial 1-2z, with root 1/2. 1/2 is inside the unit circle, $| 1/2 | \leq 1$, so the model is not causal.

c. The MA polynomial is 1, which has no roots. So it is invertible. There are no roots on or inside the unit circle.

EXAMPLE 5. Consider the causal MA(1) model $X_t = 0.7 W_{t-1} + W_t$, where W is white noise with variance 20. Let $\gamma(h)$ be the autocovariance function of X_t.

a. What is $\gamma(0)$?

b. What is $\gamma(1)$?

a. $\gamma(0) = cov(X_t , X_t) = cov(0.7 W_{t-1} + W_t, .0.7 W_{t-1} + W_t)$

$= .49\ cov(W_{t-1} , W_{t-1}) + 0 + 0 + cov(W_t , W_t)$
$= .49\ (20) + 20$

Alternatively,

$V(X_t) = V(0.7 W_{t-1} + W_t)$

$= .7^2\ V(W_{t-1}) + V(W_t) + 2\text{cov}(0.7\ W_{t-1}\ , W_t)$
$= .49\ (20) + 20 + 0.$

b. $\gamma(1) = \text{cov}(X_t\ , X_{t-1}) = \text{cov}(0.7\ W_{t-1}\ + W_t, 0.7\ W_{t-2}\ + W_{t-1})$
$= 0 + .7\ \text{cov}(W_{t-1}\ , W_{t-1}) + 0 + 0$
$= .7\ (20)$
$= 14.0.$

Here X_t is already expressed as a pure MA so the χ-weights are obvious.
Note that $\phi\ \chi\ = \theta$, so $(1)\ (1 + \chi_1 z + \chi_2 z^2 + ...) = 1 + 0.7z$.
Match the different powers of z, starting with 0.
$1 = 1.$
$\chi_1 z = 0.7z$. So $\chi_1 = 0.7$.
$\chi_2 z^2\ = 0$. So $\chi_2 = 0$. Similarly $\chi_3 = 0$, etc.
$X_t = W_t + \chi_1\ W_{t-1} + \chi_2\ W_{t-2} + ...$
$= W_t + 0.7\ W_{t-1}\ .$

EXAMPLE 6. Consider the causal, invertible, reduced ARMA(2,2) process $X_t = 0.3\ X_{t-1}\ +\ 0.1\ X_{t-2}\ +\ 0.2\ W_{t-2}\ +\ 0.4\ W_{t-1}\ + W_t$, where W is white noise with variance 20. Let $\gamma(h)$ be the autocovariance function of X_t.

a. Find the first 4 ψ-weights: ψ_1, ψ_2, ψ_3, and ψ_4.
b. Use these first 4 ψ-weights to approximate $\gamma(0)$.
c. Use these first 4 ψ-weights to approximate $\gamma(1)$.
d. What is $\text{Cov}(X_t, X_{t-3})$?
e. What is $\text{Cov}(X_t\ , W_t)$?

a. $\phi\ \chi\ = \theta$,
so $(1-.3z-.1z^2)\ (1 + \chi_1 z\ + \chi_2 z^2 + ...) = 1 + 0.4z + 0.2\ z^2$.
Be careful here! 0.4 goes with z and 0.2 goes with z^2 .
Match the different powers of z, starting with 0.
$1 = 1.$

$-.3 + \chi_1 = .4$. So $\chi_1 = .7$.
$-.1 - .3\chi_1 + \chi_2 = 0.2$. So $\chi_2 = 0.2 + .1 + .3(.7) = .51$.
$\chi_3 - .3\chi_2 - .1\chi_1 = 0$. So $\chi_3 = .1(.7) + .3(.51) = .07 + .153 = .223$.
$\chi_4 - .3\chi_3 - .1\chi_2 = 0$. So $\chi_4 = .1(.51) + .3(.223) = .1179$.

b. $X_t \sim W_t + .7\, W_{t-1} + .51\, W_{t-2} + .223\, W_{t-3} + .1179\, W_{t-4}$.

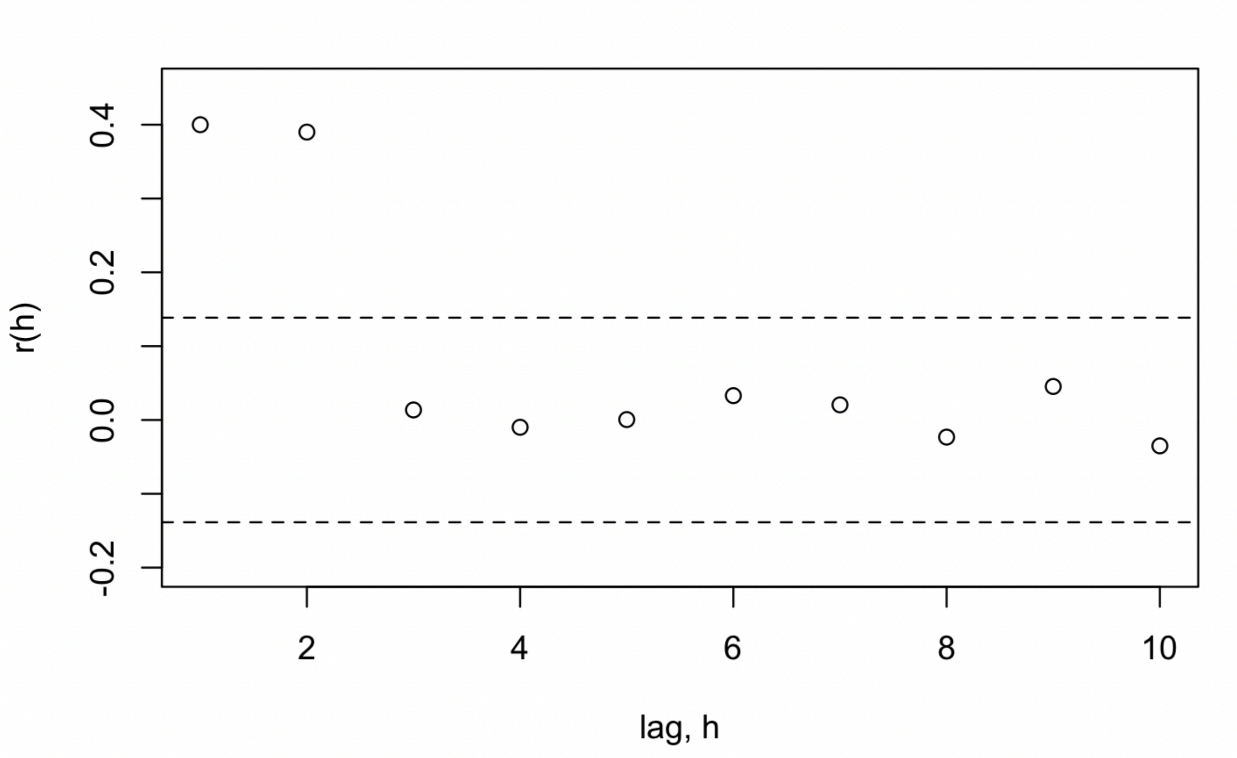
$$\begin{aligned} V(Xt) &\sim 20 + .7^2 (20) + .51^2 (20) + .223^2 (20) + .1179^2 (20) \\ &= 20\, (1 + .49 + .2601 + .049729 + .01390041) \\ &= 20\, (1.813729) \\ &= 36.27458. \end{aligned}$$

c. $\text{cov}(X_t, X_{t-1}) \sim \text{cov}(W_t + .7\, W_{t-1} + .51\, W_{t-2} + .223\, W_{t-3} + .1179\, W_{t-4}\ ,$
$W_{t-1} + .7\, W_{t-2} + .51\, W_{t-3} + .223\, W_{t-4} + .1179\, W_{t-5})$
$= 0 + .7\,(20) + (.51)(.7)\,(20) + (.223)(.51)(20) + (.1179)(.223)(20)$
$= 23.94043$.

d. $\gamma(3) = \text{Cov}(X_t, X_{t-3}) = \text{cov}(W_t + .7\, W_{t-1} + .51\, W_{t-2} + .223\, W_{t-3} + .1179\, W_{t-4}\ ,$
$W_{t-3} + .7\, W_{t-4} + .51\, W_{t-5} + .223\, W_{t-6} + .1179\, W_{t-7}\ )$
$= (.223)(1)(20) + (.1179)(.7)(20)$.

e. $\text{Cov}(X_t\ , W_t\ ) = \text{Cov}\ (0.3\, X_{t-1} + 0.1\, X_{t-2} + 0.2\, W_{t-2} + 0.4\, W_{t-1} + W_t,\ W_t)$
$= \text{cov}(W_t, W_t) = 20$.

EXAMPLE 7. Suppose the correlogram below corresponds to a time series of 200 observations. What model might be consistent with this correlogram?



MA(2).