9/16/96

Randomized Intensity Estimates for Transformed Poisson Processes

Introduction to the problem

Suppose N is a stationary Poisson process on the line with intensity λ, observed from time 0 to time T. The transformed process, M, given by $M(t) := N(t/\lambda)$, is a Poisson process with intensity 1. M may equivalently be specified by prescribing that N has a point at t iff. M has a point at $t \times \lambda$.

It was shown in [1] that if λ is estimated by maximum likelihood, the estimated transformed process $\hat{M}(t) := N(t/\hat{\lambda})$ is not a Poisson process, but instead exhibits self-correcting behavior. That is, the conditional intensity $\lambda_{\hat{m}}(t)$ of $\hat{M}$ is less than 1 when $\hat{M}(t) > t$ (i.e. when there have been more points than expected), and is greater than 1 when $\hat{M}(t) < t$ (when there have been fewer than expected). [1] also shows by simulations that, for small samples, this difference between $\hat{M}$ and a Poisson process can be rather significant.

The question addressed here is: can this problem be remedied somehow? Is there some other way to estimate λ, so that $\hat{M}$ will closer approximate the standard Poisson process?

One naive attempt may be to multiply the estimate $\hat{\lambda}$ by some factor. No such procedure can be very effective. The main problem is that for a Poisson process, the number of points in the interval [0,T] may be quite different from its expected value, $\lambda \times T$. But when λ is estimated by MLE, the transformed process $\hat{M}$ is guaranteed to contain $N(T)$ points in a region of size $N(T)$. Multiplying $\hat{\lambda}$ by some factor simply forces $N(T)$ to be that factor of the size of the transformed region, and does not allow the ratio of the number of points to the size of the transformed region to vary randomly in the way that the ratio $N(T)/T$ varies for the standard Poisson process.

Randomized Bayes estimates

Another approach is to let the estimate $\hat{\lambda}$ be randomized. That is, conditional on the observation of the point process N on $[0, T]$, generate a random variable $\hat{\lambda}$ which may depend on N.

One way to do this involves a Bayesian approach. Place a uniform prior on R^+ for λ, and for x in R^+, let $P(\hat{\lambda} = x|N(T) = i)$ be equal to the posterior probability $P(\lambda = x|N = i)$.

It works out then that

$$P(\hat{\lambda} = x|N = i) \quad = \quad \frac{P(N = i|\lambda = x)}{\int_0^\infty P(N = i|\lambda = y)dy}$$

$$= \quad exp(-x)x^i/i!.$$

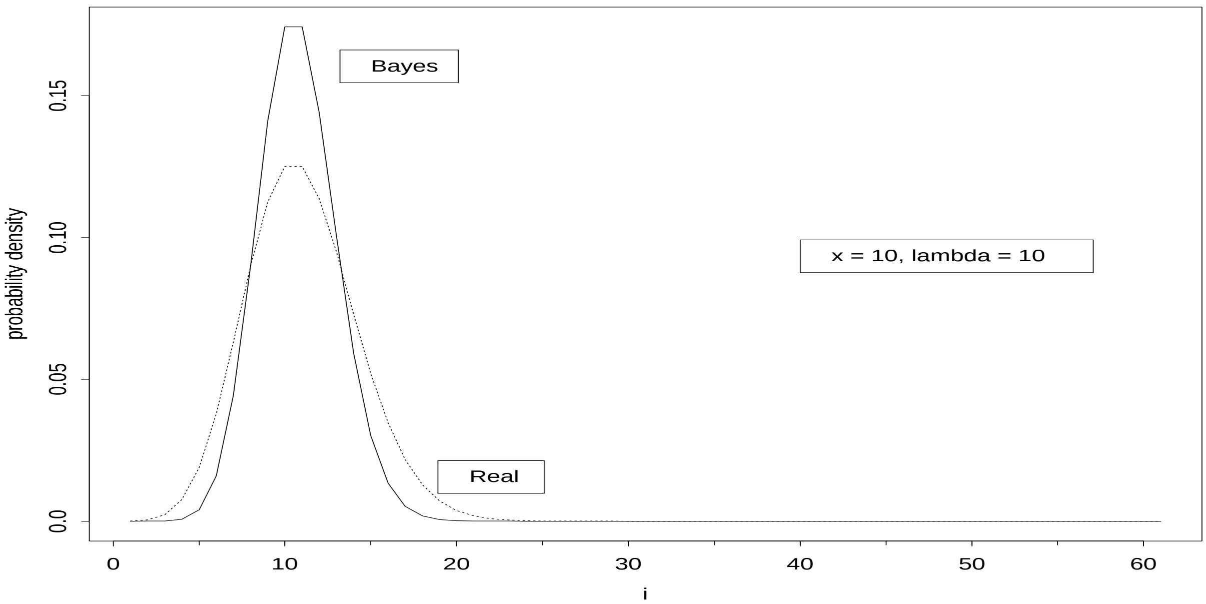
Now, ideally one would like the estimated transformed process to behave like a Poisson process, so one would want $P(N = i|\hat{\lambda} = x)$, which is the probability that the transformed process contains i points given that it has length x, to equal the probability that a Poisson random variable with mean x takes the value i, i.e.,

$$exp(-x)x^i/i!. \tag{0.1}$$

Using Bayes formula, it is not hard to show that

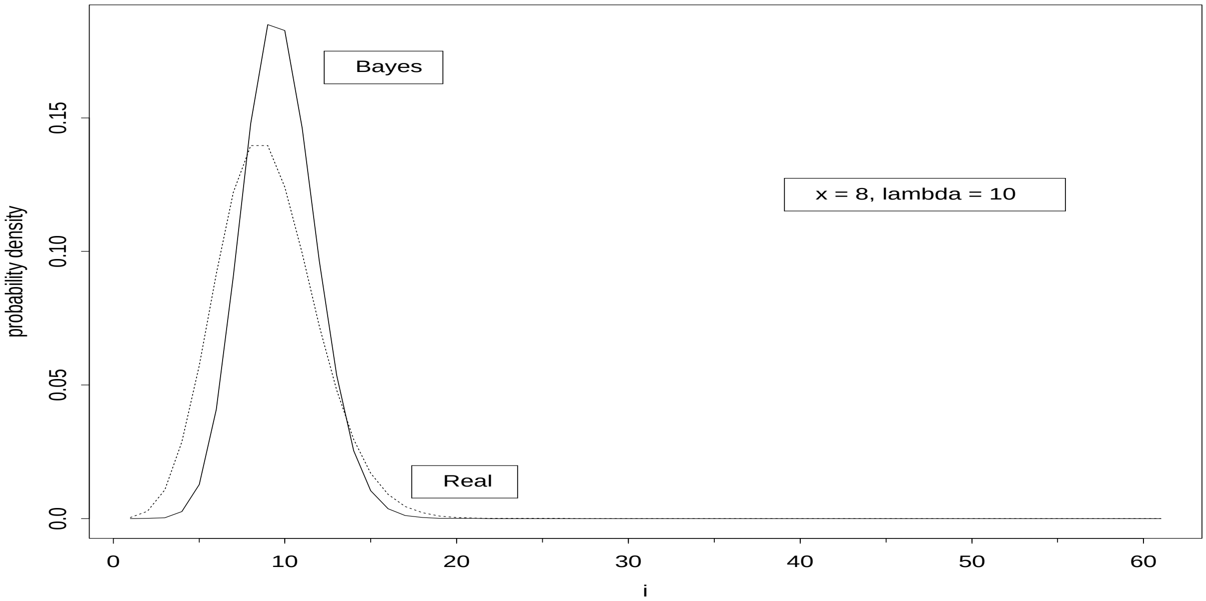
$$P(N = i|\hat{\lambda} = x) = \frac{(x\lambda)^i/(i!)^2}{\sum_{j=0}^{\infty}(x\lambda)^j/(j!)^2}. \tag{0.2}$$

Figure 0.1: $P(N = i|\hat{\lambda}_{BAYES} = x)$ and $P(N = i)$ vs. i



One question is whether formula (0.2) approximates formula (0.1) reasonably well, for most relevant values of k, λ, and i. This question is addressed in Figures 0.1 to 0.4. In these Figures, different values of λ and x are selected, and the "real" (Poisson) distribution of N is compared to the conditional distribution of N given $\hat{\lambda}$.

Figure 0.2: $P(N=i|\hat{\lambda}_{BAYES}=x)$ and $P(N=i)$ vs. i



It seems that the approximation is ok, but not great. The two curves are fairly close together, but the conditional distribution is too narrow in every case; i.e the distribution is too tightly peaked around λ. To somehow widen the distribution would appear to be appropriate.

Poisson estimates

Instead of the Bayes estimate, another approach could be the following. Given $N(T)=i$, draw a Poisson random variable with mean i, and let that be the estimate of λ. In other words, let $P(\hat{\lambda}=x|N=i)=exp(-i)i^x/\Gamma(x+1)$. (Here, the continuous analog of the Poisson random variable seems appropriate, since λ is not restricted to integer values.)

Again, we would like $P(N=i|\hat{\lambda}=x)$ to approximate (for all λ) $exp(-x)x^i/i!$.

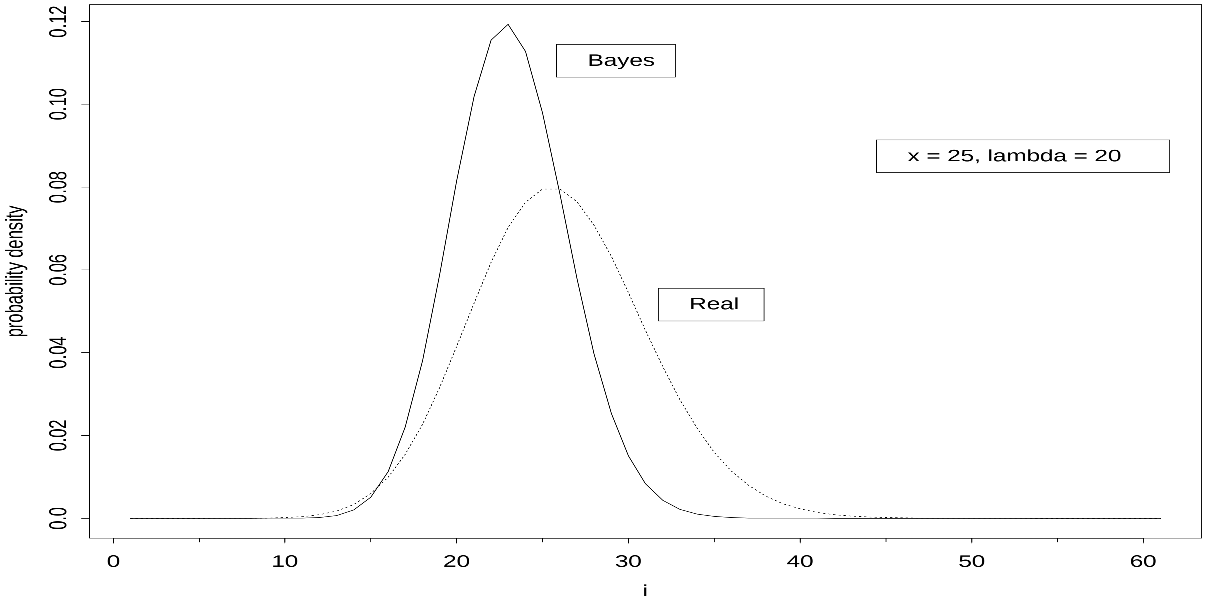
$$\begin{aligned} P(N=i|\hat{\lambda}=x) &= P(\hat{\lambda}=x|N=i)\times P(N=i)\div P(\hat{\lambda}=x) && (0.3)\\ &= P(\hat{\lambda}=x|N=i)P(N=i)\div\sum_{j=0}^{\infty}P(\hat{\lambda}=x|N=j)\times P(N=j). && (0.4)\\ & && (0.5)\end{aligned}$$

The numerator in (0.4) equals $exp(-i)i^x/\Gamma(k+1)\times exp(-\lambda)\lambda^i/i!$.

The denominator in (0.4) equals $\sum_j exp(-j)j^x/\Gamma(k+1)\times exp(-\lambda)\lambda^j/j!$.

So, the ratio in (0.4) may be written:

Figure 0.3: $P(N = i|\hat{\lambda}_{BAYES} = x)$ and $P(N = i)$ vs. i



$$exp(-i)i^x\lambda^i/i! \div \sum_j exp(-j)j^x\lambda^j/j!.$$

Figure 0.5 shows a comparison of the conditional and unconditional distributions of N when values λ and x are selected as in Figure 0.1, where now the estimate $\hat{\lambda}$ is made using the Poisson estimate. The results are virtually the same as with the Bayes estimate; again, the conditional distribution appears to be too narrow. The results for the Poisson estimate appear to be the same as for the Bayes estimate for all choices of λ and x, though only the first of the four plots is shown here.

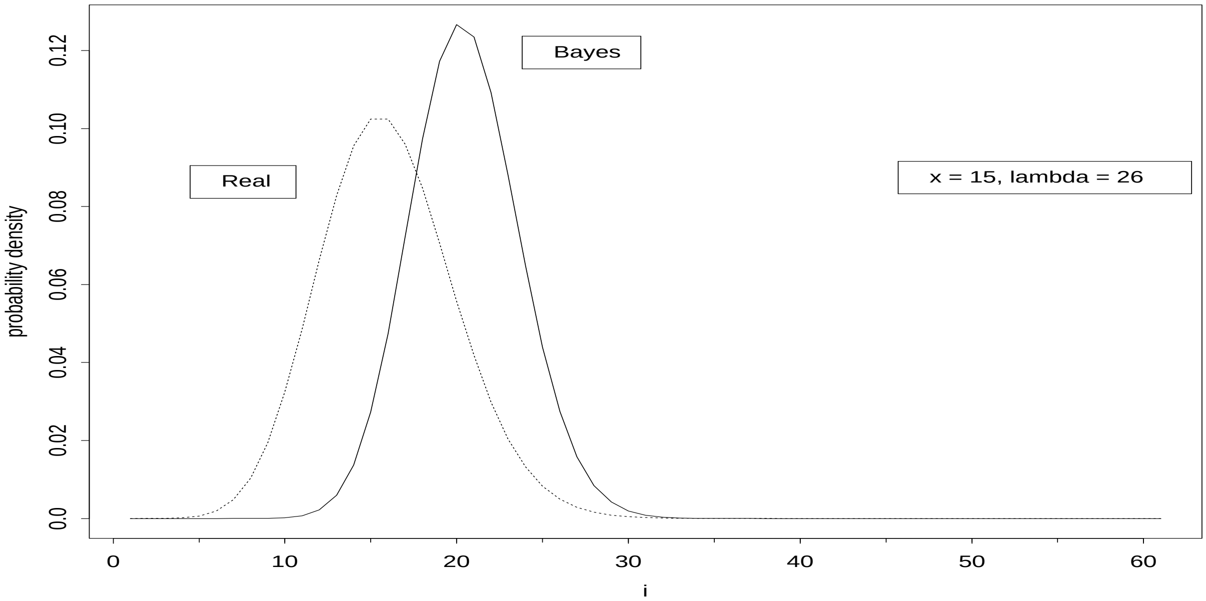
The difference between the Bayes estimate of Figure 0.1 and the Poisson estimate of Figure 0.5 is displayed in Figure 0.6. While there is a noticeable difference between the two conditional densities in Figure 0.6, the difference is hardly noticeable in terms of the scale of Figures 0.1 and 0.5.

Gamma estimates

One effort to widen the conditional distribution of N given $\hat{\lambda}$ is to select $\hat{\lambda}$ from a distribution less peaked than the Poisson. For instance, one option is to let the conditional distribution of $\hat{\lambda}$ given $N = i$ follow a Gamma distribution, with scale parameter i (provided $i > 0$).

When the Gamma estimate is used, the situation is identical to that of the Bayes and Poisson estimates. Comparing Figure 0.7 with Figures 0.1 and 0.5, one sees that in all three

Figure 0.4: $P(N = i|\hat{\lambda}_{BAYES} = x)$ and $P(N = i)$ vs. i



cases, the conditional distribution of N appears to be too tightly peaked around its mean. Again, this situation remains the same when different choices of λ and x are used.

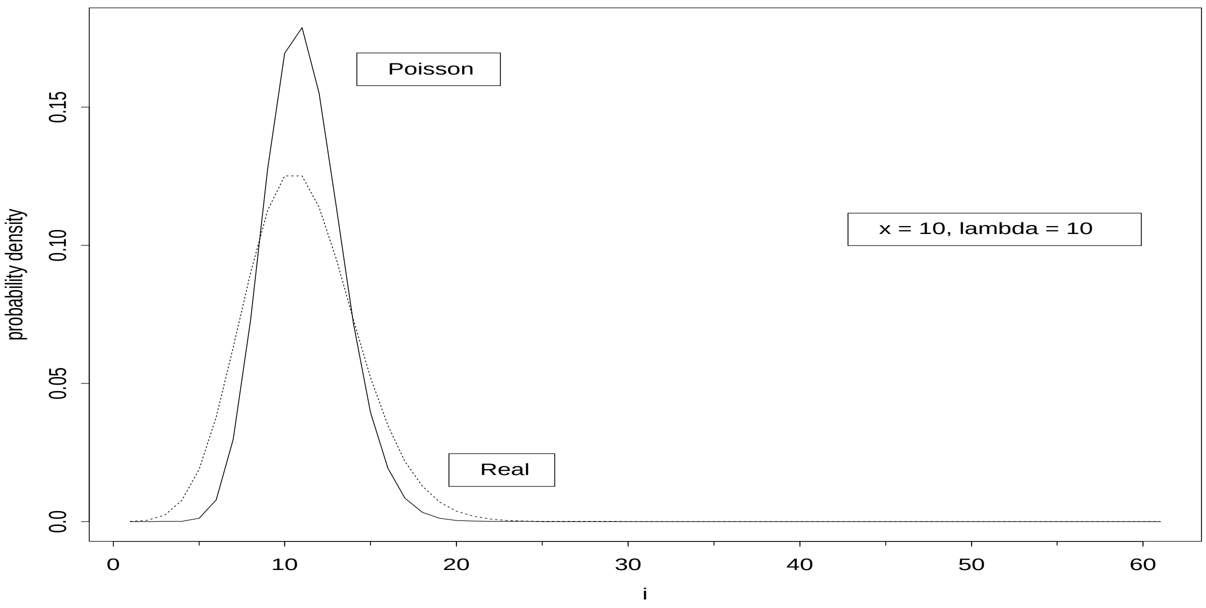
Simulations

On the other hand, the fact that the conditional distribution of N given $\hat{\lambda}$ is different from the distribution of N does not necessarily mean that the transformed processes based on the Bayes, Poisson, or Gamma estimates are necessarily much different from standard Poisson processes. One way to check how different the resulting processes are is by simulation.

Consider the ratio $R = N(T)/EN(T)$. For the standard Poisson process observed on an interval of length T, this ratio has mean 1 and variance $1/T$. If the Gamma-estimated and transformed process is similar to the standard Poisson process, then the distribution of R for the transformed processes should be similar to that of the Poisson process. Figure 0.8 shows the histogram of the values of R for 200 standard Poisson processes observed on intervals of length 10. By comparison, Figure 0.9 shows the histograms of the values of R for 200 transformed Poisson processes of intensity 10 observed on intervals of length 1 whose intensity estimates were selected at random, conditional on $N(T)$, using the Gamma distribution. The distributions are evidently rather close, but there are some noticeable differences. The distribution of R for the Gamma-estimated processes appears to be more strongly skewed to the right. Note that, if instead the intensities were estimated by maximum likelihood, then the ratio R would be identically 1 for every single simulation. Thus, clearly the randomized estimation provides some improvement in this regard.

It should be noted that the distributions of such statistics as the normalized TTT-statistic and the normalized Kolmogorov-Smirnov statistic are invariant to the problem of parameter

Figure 0.5: $P(N = i|\hat{\lambda}_{Poisson} = x)$ and $P(N = i)$ vs. i



estimation when the point process being estimated is a stationary Poisson process. Because of the normalization, the statistic is completely independent of the parameter estimate. However, the same is not necessarily true when N is not a stationary Poisson process, and even for the stationary Poisson case, estimation may affect the distributions of some nonnormalized test statistics and some other statistics, e.g. the second order statistics in Heinrich (1991).

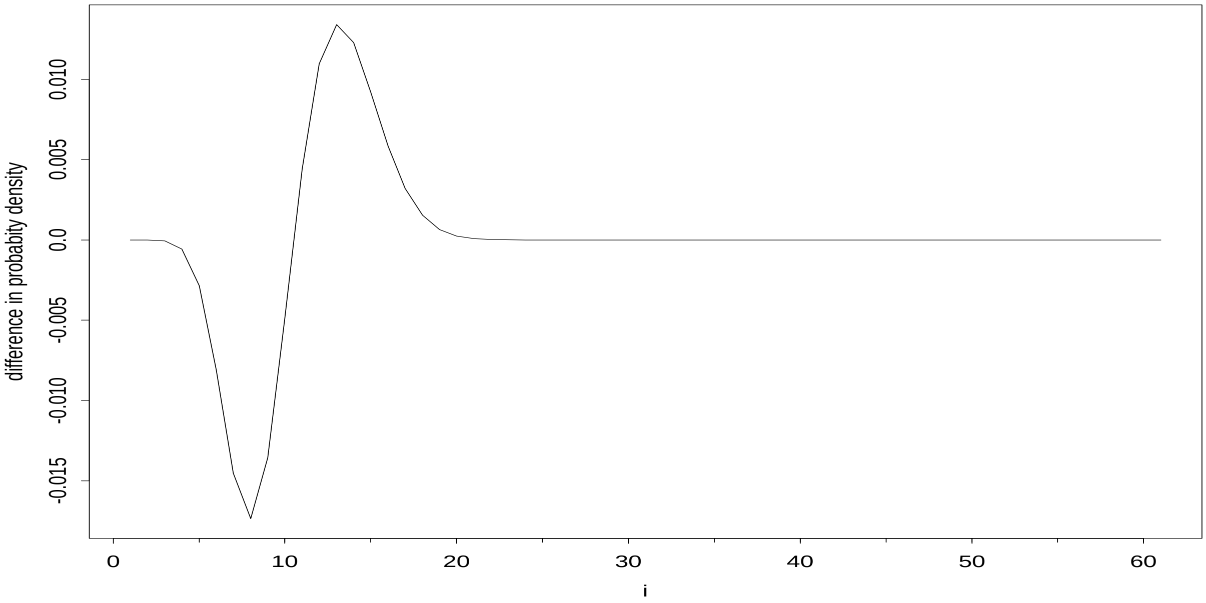
Conclusion

When the rate parameter is estimated by maximum likelihood, the transformed process differs from the standard Poisson process, and these differences cannot be remedied by multiplying parameter estimates by a scaling factor. However, when randomness is introduced and the estimates are made at random, conditional on the observation of the point process from time 0 to time T, the transformed process appears to bear an increased resemblance to the standard Poisson process. In particular, the ratio of total number of observed points to expected number of observed points changes from a point mass at 1 to a distribution resembling that of the ratio for the standard Poisson process.

The three randomized estimates proposed here: Bayes, Poisson, and Gamma estimates, all appear to behave about the same as one another, and all provide improvement over maximum likelihood estimates with respect to certain specific properties. Further statistical properties of processes transformed via randomized intensity estimates, such as distributions of standard Poisson test statistics, may be a subject for future research.

References

Figure 0.6: $P(N = i|\hat{\lambda}_{Poisson} = x) - P(N = i|\hat{\lambda}_{BAYES} = x)$ vs. i



[1] Schoenberg, F. A naturally arising self-correcting point process.

Figure 0.7: $P(N = i|\hat{\lambda}_{Poisson} = x) - P(N = i|\hat{\lambda}_{BAYES} = x)$ vs. i

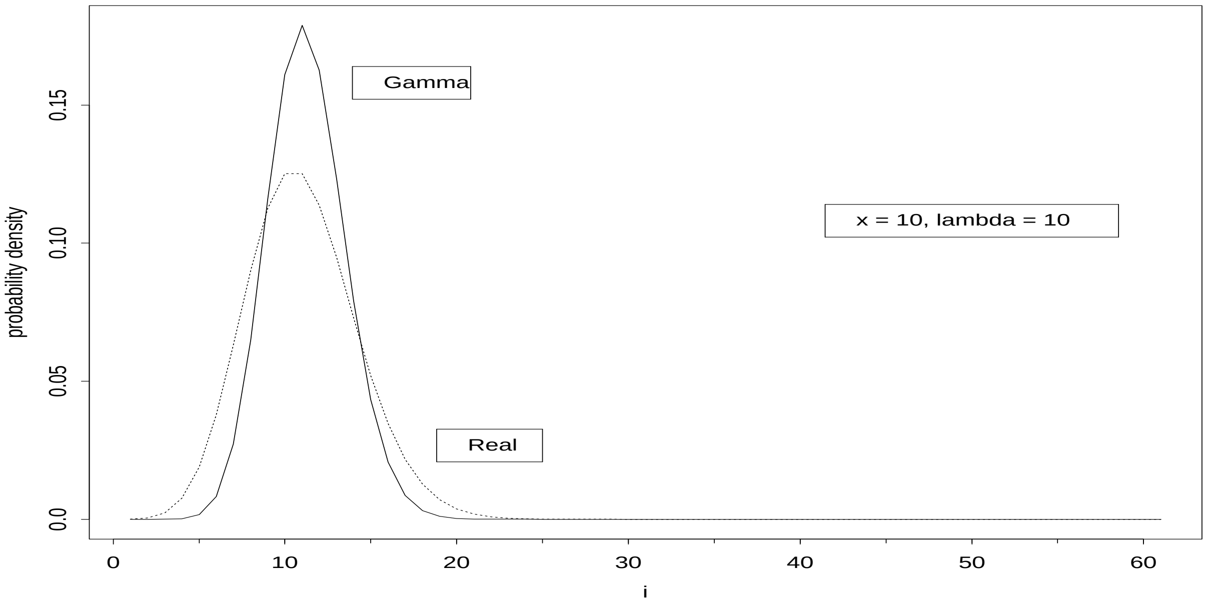


Figure 0.8: Histogram of R for 200 standard Poisson processes

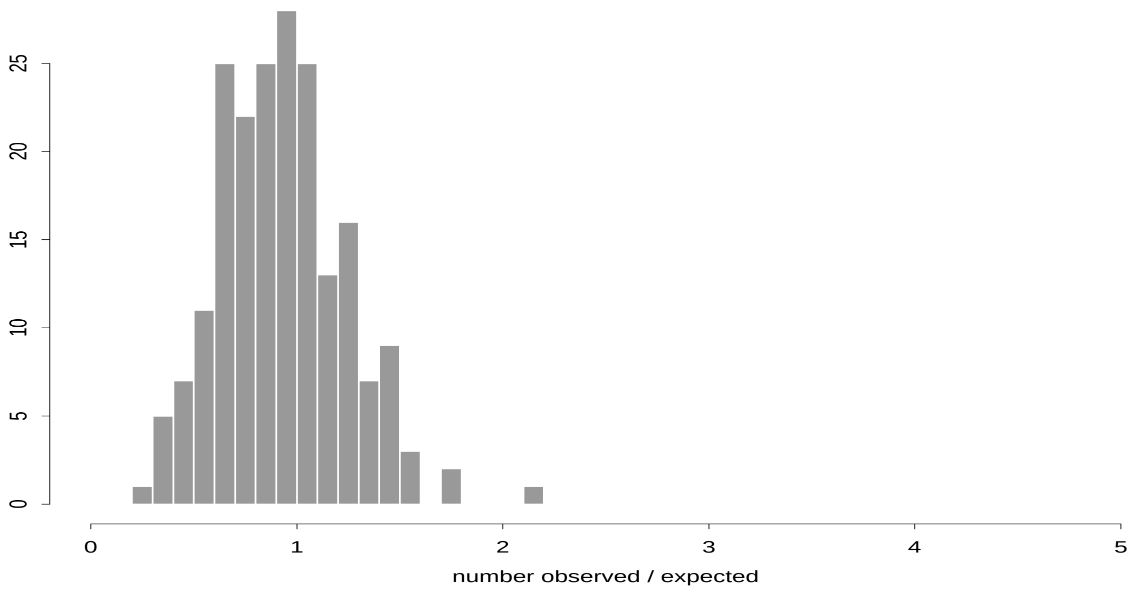


Figure 0.9: Histogram of R for 200 Gamma-transformed processes