On Rescaled Poisson Processes and the Brownian Bridge *

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Abstract

The process obtained by rescaling a homogeneous Poisson process by the maximum likelihood estimate of its intensity is shown to have surprisingly strong self-correcting behavior. Formulas for the conditional intensity and moments of the rescaled Poisson process are derived, and its behavior is demonstrated using simulations. Relationships to the Brownian Bridge are explored, and implications for point process residual analysis are discussed.

Keywords: Brownian bridge; Poisson bridge; intensity; point process; Poisson process; residual analysis

1 Introduction.

The random time change theorem dictates how one may rescale a point process N in order to obtain a Poisson process with unit intensity (Meyer, 1971; Papangelou, 1972; Brémaud, 1972). The procedure amounts to stretching or compressing the point process according to its conditional intensity process, λ. For instance if N is a stationary Poisson process on the line with constant intensity $\lambda > 0$, the rescaled process M defined via

$$M(a, b) := N(a/\lambda, b/\lambda) \tag{1.1}$$

is a Poisson process of unit rate. The random time change theorem applies to any simple point process on the line (Meyer, 1971), and has been extended to wide classes of point processes in higher dimensions (Merzbach and Nualart, 1986; Nair, 1990; Schoenberg, 1999)).

The above results all require that the conditional intensity λ of the point process be known. Thus one may question whether the rescaled process is similar to a Poisson process when λ is estimated rather than known. The present paper investigates the behavior of the rescaled process $\hat{M}_T$ obtained by rescaling N according to $\hat{\lambda}_T$, the maximum likelihood estimate of λ, for the case where N is a stationary Poisson process on the line observed from time 0 to time T. In such cases, the rescaled process is found to be quite different from the Poisson process with unit rate.

In practice, analysis of the rescaled process $\hat{M}_T$ is often used in so-called point process *residual analysis*; applications include model evaluation (Schoenberg, 1997) and point process prediction (Ogata, 1988). The fact that $\hat{M}_T$ is not a Poisson process, or equivalently that $N(0,t) - \hat{\lambda}_T t$ is not a martingale, has been observed by several authors including Aalen and Hoem (1978), Brown and Nair (1988), Heinrich (1991), and Solow (1993). It has been argued that the difference between $\hat{M}_T$ and a unit-rate Poisson process is negligible, since $\hat{\lambda}_T$ converges a.s. to λ, or because certain statistics, such as the Kolmogorov-Smirnov statistic, when applied to $\hat{M}_T$ have asymptotically the same distribution as the statistics corresponding to the Poisson process (Saw, 1975; Davies, 1977; Kutoyants, 1984; Ogata and Vere-Jones, 1984; Lisek and Lisek, 1985; Lee, 1986; Arsham, 1987; Karr, 1991; Heinrich, 1991; Yokoyama et al., 1993). Therefore until now the properties distinguishing $\hat{M}_T$ from the unit-rate Poisson process have not been extensively investigated.

However, when T is small, the asymptotic arguments above are less relevant, and the exact properties of $\hat{M}_T$ may be important. The current paper demonstrates the self-correcting nature of $\hat{M}_T$ and its highly fluctuating conditional intensity process. This suggests that caution should be used in assuming that $\hat{M}_T$ is similar to the Poisson process, particularly when the original point process N is observed over a short time scale. Further, although certain functionals of $\hat{M}_T$ may asymptotically approach those of a Poisson process, the asymptotic behavior of $\hat{M}_T$ may alternatively be chacterized in relation to the Brownian Bridge.

The structure of this paper is as follows. Section 2 lists a few definitions and conventions dealing with notation. Finite-sample properties of rescaled Poisson processes are investigated in Section 3. Section 4 presents results related to the asymptotic properties of $\hat{M}_T$. In Section 5, the extent of the self-correcting behavior of $\hat{M}_T$ is demonstrated using simulations.

2 Preliminaries

Throughout this paper we will let N refer to a homogeneous Poisson process on the real half-line $\mathbf{R}_+$ with intensity $\lambda > 0$, observed from time 0 to time T, and M will denote the rescaled Poisson process defined by relation (1.1). Thus N has points at times $\tau_1, \tau_2, \ldots, \tau_n$ iff. M has points at times $\tau_1/\lambda, \tau_2/\lambda, \ldots, \tau_n/\lambda$.

The definitions that follow relate to an arbitrary point process P on the real half-line $\mathbf{R}_+$. We say P is *self-correcting* if, for $0 < a \leq b < c$,

$$\text{cov}\{P(a,b), P(b,c)\} < 0,$$

and P is called *self-exciting* if this covariance is positive. Well-known examples of self-correcting and self-exciting point processes, respectively, are the stress-release process (Isham and Westcott, 1979) and the Hawkes process (Hawkes, 1971). Note that the term self-exciting is sometimes used to refer exclusively to the process described by Hawkes, rather than the more general meaning given here.

When it exists, the *conditional intensity* process λ associated with P is defined by

$$\lambda(t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathrm{E}\left[P[t, t+\Delta t)|H_t\right],$$

where H_t is the filtration generated by $P[0,t)$ from time 0 to time t. It is well known that when it exists, the conditional intensity is unique a.e. and determines all the finite-dimensional distributions of P (Daley and Vere-Jones, 1988). Therefore a natural way to characterize a point process is via its conditional intensity. Note that for the stationary Poisson process N, the conditional intensity λ is constant a.e. and N is neither self-correcting nor self-exciting.

Following convention, we abbreviate the random variable $P[0,t]$ by $P(t)$. Thus $P(t)$ is the P-measure of the interval $[0,t]$; it is important to distinguish this from $P(\{t\})$, i.e. the measure P assigns to the point $\{t\}$.

Let n denote the total number of observed points $N(T)$. $\hat{\lambda}_T = n/T$, the maximum likelihood estimate of λ. Assuming $\hat{\lambda}_T > 0$, let $\hat{M}_T$ denote the point process defined via:

$$\hat{M}_T(a,b) := N(a/\hat{\lambda}_T, b/\hat{\lambda}_T),$$

for $0 \le a \le b \le n$. That is, $\hat{M}_T$ is the process with points at times $\tau_1/\hat{\lambda}_T, \tau_2/\hat{\lambda}_T, \ldots, \tau_n/\hat{\lambda}_T$. In the case that $\hat{\lambda}_T = 0$, let $\hat{M}_T(a,b) = 0$ for all $a, b > 0$. Similarly, set $\hat{M}_T(a,b) = 0$ if $n < a \le b$.

The following function arises repeatedly in calculations of finite-sample properties of $\hat{M}$. Let

$$\phi(x,t) := \frac{exp(-x)}{S(x,t)} \left[Ei(x) - \gamma - ln(x) - \sum_{i=1}^{\lfloor t \rfloor} \frac{x^i}{i \times i!} \right],$$

where $S(x,t)$ is the survivor function for a Poisson random variable with mean x:

$$S(x,t) := 1 - exp(-x) \sum_{i=0}^{\lfloor t \rfloor} x^i/i!\,,$$

$Ei(x)$ is the standard exponential integral, defined as the Cauchy principal value of the integral

$$Ei(x) := \oint_{-\infty}^{x} exp(u)/u \; du\,,$$

and γ is Euler's constant:

$$\gamma := \lim_{J \to \infty} \{ \sum_{j=1}^{J} 1/j - ln(J) \} \approx .5772...$$

Finally, let **I** denote the indicator function.

3 Finite-sample characteristics of $\hat{M}_T$

The support of $\hat{M}_T$ is the subset $[0, n]$, which is random. In residual analysis of point processes, one is interested only in the behavior of $\hat{M}_T$ within this support. Thus, of particular concern are the *conditional* moments of $\hat{M}_T(a, b)$, given that $n \geq b$. Such properties are given in Theorem 3.1 below.

THEOREM 3.1. For $0 \leq a \leq b \leq c$,

(i) $\mathrm{E}[\hat{M}_T(a, b) | n \geq b] = b - a$.

(ii) $\mathrm{Cov}\{\hat{M}_T(a, b)\ ,\ \hat{M}_T(b, c) | n \geq c\} = -(b - a)(c - b)\phi(\lambda T, c)$.

(iii) $\mathrm{Var}\{\hat{M}_T(a, b) | n \geq b\} = b - a - (b - a)^2 \phi(\lambda T, b)$.

PROOF. (i) Given that $n = k$, the unordered points $\{\tau_1, \tau_2, ..., \tau_k\}$ of N

are i.i.d. uniform random variables on $[0, T]$. Hence

$$\begin{aligned}
\mathrm{E}[\hat{M}_T(a,b)|n \geq b] &= \sum_{k=\lceil b \rceil}^{\infty} \mathrm{E}[\hat{M}_T(a,b)|n = k]\, P\{n = k|n \geq b\} \\
&= \sum_{k=\lceil b \rceil}^{\infty} \mathrm{E}[N(aT/k, bT/k)|n = k]\, P\{n = k|n \geq b\} \\
&= \sum_{k=\lceil b \rceil}^{\infty} k(bT/k - aT/k)/T\, P\{n = k|n \geq b\} \\
&= (b-a) \sum_{k=\lceil b \rceil}^{\infty} P\{n = k|n \geq b\} \\
&= b - a.
\end{aligned}$$

(ii) Note that

$$\begin{aligned}
\sum_{k=\lceil c \rceil}^{\infty} P\{n = k|n \geq c\}/k &= \frac{exp(-\lambda T)}{P\{n \geq c\}} \sum_{k=\lceil c \rceil}^{\infty} \frac{(\lambda T)^k}{(k \times k!)} \qquad (3.1) \\
&= \phi(\lambda T, c)
\end{aligned}$$

the last relation following from equation (5.1.10) of Abramowitz (1964).

Conditioning again on n, and letting $p_{k,c}$ denote $P\{n = k|n \geq c\}$ we may write

$$\begin{aligned}
&\mathrm{E}[\hat{M}_T(a,b)\ \hat{M}_T(b,c)|n \geq c] \\
&\quad = \sum_{k=\lceil c \rceil}^{\infty} \mathrm{E}[\hat{M}_T(a,b)\hat{M}_T(b,c)|n = k]\, p_{k,c}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=\lceil c\rceil}^{\infty} \mathrm{E}[N(aT/k,bT/k)N(bT/k,cT/k)|n=k]\, p_{k,c} \\
&= \sum_{k=\lceil c\rceil}^{\infty} \mathrm{E}[\sum_{i=1}^{k} \mathbf{I}\{\tau_i \in (aT/k,bT/k)\} \sum_{j=1}^{k} \mathbf{I}\{\tau_j \in (bT/k,cT/k)\}|n=k]\, p_{k,c} \\
&= \sum_{k=\lceil c\rceil}^{\infty} \mathrm{E}[\sum_{i\neq j} \mathbf{I}\{\tau_i \in (aT/k,bT/k)\}\mathbf{I}\{\tau_j \in (bT/k,cT/k)\}|n=k]\, p_{k,c} \\
&= \sum_{k=\lceil c\rceil}^{\infty} (k^2-k)\mathrm{E}[\mathbf{I}\{\tau_1 \in (aT/k,bT/k)\}\mathbf{I}\{\tau_2 \in (bT/k,cT/k)\}|n=k]\, p_{k,c} \\
&= \sum_{k=\lceil c\rceil}^{\infty} (k^2-k)\frac{(b-a)}{k}\,\frac{(c-b)}{k} p_{k,c} \\
&= (b-a)(c-b)\left[1-\sum_{k=\lceil c\rceil}^{\infty} p_{k,c}/k\right],
\end{aligned}
$$

which along with (i) and (3.1) establishes (ii).

(iii) Similarly,

$\mathrm{E}[\hat{M}_T(a,b)^2\,|n\geq b]$

$$
\begin{aligned}
&= \sum_{k=\lceil b\rceil}^{\infty} \mathrm{E}[N(aT/k,bT/k)N(aT/k,bT/k)|n=k]\, p_{k,b} \\
&= \sum_{k=\lceil b\rceil}^{\infty} \mathrm{E}[\sum_{i=1}^{k} \mathbf{I}\{\tau_i \in (aT/k,bT/k)\} \sum_{j=1}^{k} \mathbf{I}\{\tau_j \in (aT/k,bT/k)\}|n=k]\, p_{k,b} \\
&= \sum_{k=\lceil b\rceil}^{\infty} \mathrm{E}[\sum_{i\neq j} \mathbf{I}\{\tau_i \in (aT/k,bT/k)\}\mathbf{I}\{\tau_j \in (aT/k,bT/k)\} \\
&\qquad + \sum_{i=1}^{k} \mathbf{I}\{\tau_i \in (aT/k,bT/k)\}|n=k] \times\, p_{k,b} \\
&= \sum_{k=\lceil b\rceil}^{\infty} \left[(k^2-k)\left(\frac{b-a}{k}\right)^2 + \frac{k(b-a)}{k}\right] p_{k,b}
\end{aligned}
$$

$$
\begin{aligned}
&= b - a + (b-a)^2 \left[1 - \sum_{k=\lceil b \rceil}^{\infty} P\{n = k | n \geq b\}/k \right] \\
&= b - a + (b-a)^2 [1 - \phi(\lambda T, b)].
\end{aligned}
$$

REMARK 3.2. It is evident from (3.1) that $\phi(\lambda T, c)$ is positive. Thus equation (ii) of Theorem 3.1 implies that

$$
\mathrm{Cov}\{\hat{M}_T(a,b), \hat{M}_T(b,c) | n \geq c\} < 0,
$$

i.e. $\hat{M}_T$ is a self-correcting point process.

Fix $t > 0$. Let m denote $\hat{M}_T(t)$. Let $z := \lceil t^+ \vee m \rceil$, i.e. the least integer strictly greater than t and greater than or equal to m. Let $z' := \lceil t^+ \vee (m+1) \rceil$. Although no closed form is available for the conditional intensity of $\hat{M}_T$, a formula which is useful in practice for calculating an approximation is given in the following result.

THEOREM 3.3. The conditional intensity process $\lambda_{\hat{M}}$ corresponding to the point process $\hat{M}_T$ satisfies:

$$
\lambda_{\hat{M}}(t) = \sum_{k=z'}^{\infty} \frac{(\lambda T)^k (k-t)^{k-m-1}}{(k-m-1)! k^k} \left[\sum_{k=z}^{\infty} \frac{(\lambda T)^k (k-t)^{k-m}}{(k-m)! k^k} \right]^{-1}. \tag{3.2}
$$

PROOF.

Let $\hat{H}_t$ denote the history of $\hat{M}_T$ from time 0 to t, i.e. the σ-field generated by $\{\hat{M}_T(x); 0 \leq x < t\}$.

$$\begin{aligned}\lambda_{\hat{M}}(t) &= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathrm{E}[\hat{M}_T[t, t+\Delta t)|\hat{H}_t] \\ &= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \sum_{k=z}^{\infty} \mathrm{E}\left[\hat{M}_T[t, t+\Delta t)|\hat{H}_t; n=k\right] P\left\{n=k|\hat{H}_t\right\} \\ &= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \sum_{k=z}^{\infty} \mathrm{E}\left[N[tT/k, (t+\Delta t)tT/k)|\hat{H}_t; n=k\right] P\left\{n=k|\hat{H}_t\right\} \end{aligned} \tag{3.3}$$

Conditional on $\hat{H}_t$ and on $n = k$, there are $k - m$ points left to be distributed by the point process N between time tT/k and time T. Since N is a Poisson process, these additional points are uniformly distributed on $[tT/k, T]$. Thus

$$\begin{aligned}\mathrm{E}\left[N[tT/k, (t+\Delta t)tT/k)|\hat{H}_t; n=k\right] &= (k-m) \ \frac{\Delta t T/k}{T - tT/k} \\ &= \frac{\Delta t(k-m)}{k-t}. \end{aligned} \tag{3.4}$$

Putting together (3.3) and (3.4) yields:

$$\lambda_{\hat{M}}(t) = \sum_{k=z}^{\infty} \frac{k-m}{k-t} P\{n=k|\hat{H}_t\}. \tag{3.5}$$

Note that $P\{n = k \mid \hat{H}_t\} = P\{n = k \mid \hat{M}(t)\}$. This relation follows from the fact that N is a Poisson process and therefore for any k, conditional on

$\{n = k; \hat{M}_T(t) = m\}$, the m points falling between time 0 and time tT/k of the process N are uniformly distributed on $[0, tT/k]$. Using Bayes' formula,

$$\begin{aligned}
P\{n = k|\hat{H}_t\} &= P\{n = k|\hat{M}_T(t) = m\} \\
&= \frac{P\{\hat{M}_T(t) = m|n = k\}P\{n = k\}}{\sum_k P\{\hat{M}_T(t) = m|n = k\}P\{n = k\}} \\
&= \binom{k}{m}\left(\frac{t}{k}\right)^m \left(1 - \frac{t}{k}\right)^{k-m} e^{-\lambda T}\frac{(\lambda T)^k}{k!} \\
&\quad \times \left[\sum_{k=z}^{\infty} \binom{k}{m}\left(\frac{t}{k}\right)^m \left(1 - \frac{t}{k}\right)^{k-m} e^{-\lambda T}\frac{(\lambda T)^k}{k!}\right]^{-1} \\
&= \frac{(\lambda T)^k (k-t)^{k-m}}{(k-m)!k^k}\left[\sum_{k=z}^{\infty} \frac{(\lambda T)^k (k-t)^{k-m}}{(k-m)!k^k}\right]^{-1},
\end{aligned}$$

which together with (3.5) yields the desired result.

4 Asymptotic properties of $\hat{M}$

It is well known that the asymptotic behavior of the normalized Poisson process can be expressed in terms of the Brownian Bridge. Both the Brownian bridge process and the rescaled Poisson process, or Poisson bridge, have been studied in connection with empirical distributions; see Major (1990) or Csörgő and Horváth (1992) for recent results. In this context, Kac (1949) showed that the distribution of the supremum of the rescaled Poisson process converges to that of the Brownian bridge. This result has been extended to show that the rescaled Poisson process, normalized as in the following result, converges weakly in the Skorohod space $D[0,1]$ to the Brownian bridge.

Theorem 4.1.

$$\left\{\frac{N(sT)-sn}{\sqrt{\lambda T}}; 0 \leq s \leq 1\right\} \Rightarrow \left\{B^0(s); 0 \leq s \leq 1\right\}$$

as $T \to \infty$, where $B^0(s)$ denotes the Brownian Bridge process on $[0,1]$.

Versions of Theorem 4.1 have been proven by various authors; see e.g. Bretagnolle and Massart (1989), who show in their Theorem 1 that furthermore an upper bound for the rate of convergence is $\log(T)/\sqrt{T}$.

Of concern in the present work is the asymptotic behavior of $\hat{M}_T$, observed from time 0 to the random time n. Let

$$X_T(s) \ := \ \frac{\hat{M}_T(sn) - sn}{\sqrt{n}}.$$

The connection between $\hat{M}_T$ and the Brownian Bridge is summarized in the following result, which is a simple extension of Theorem 4.1.

COROLLARY 4.2.

$$\{X_T(s); 0 \le s \le 1\} \Rightarrow \left\{B^0(s); 0 \le s \le 1\right\}.$$

PROOF. From Theorem 4.1 and Slutsky's theorem it is sufficient to prove that the difference between X_T and $\left\{\frac{N(sT)-sn}{\sqrt{\lambda T}}\right\}$ converges to zero in probability in the Skorohod space $D[0,1]$, which is in turn implied by convergence to zero in probability using the uniform metric on $D[0,1]$ (see e.g. chapter 18 of Billingsley, 1968, or chapter 4 of Pollard, 1984). In light of the fact that X_T can be rewritten as

$$X_T(s) = \frac{N(sT) - sn}{\sqrt{n}},$$

all that is required to complete the proof is Lemma 4.3 below.

LEMMA 4.3.

$$\underset{0 \le s \le 1}{sup} \left| \frac{N(sT) - sn}{\sqrt{\lambda T}} - \frac{N(sT) - sn}{\sqrt{n}} \right| \underset{p}{\rightarrow} 0.$$

PROOF. Let

$$d_T := \underset{0 \le s \le 1}{sup} \left| N(sT) - sn \right|.$$

Choose any positive ϵ and δ. From Theorem 4.1, $d_T/\sqrt{\lambda T} \Rightarrow \underset{0 \le s \le 1}{sup} \left| B^0(s) \right|$

as $T \to \infty$. It follows that we may find constants c and T', so that for $T > T'$,

$$P\left(d_T/\sqrt{\lambda T} \geq c\right) \leq \delta/3.$$

Let $k = c/(c-\epsilon) - 1 > 0$.

For $T \geq T'$,

$$\begin{aligned}
P\left(\frac{d_T}{\sqrt{\lambda T}} - \frac{d_T}{\sqrt{n}} > \epsilon\right) &\leq P(d_T > c\sqrt{\lambda T}) + P\left(\frac{c\sqrt{\lambda T}}{\sqrt{\lambda T}} - \frac{c\sqrt{\lambda T}}{\sqrt{n}} > \epsilon\right) \\
&\leq \delta/3 + P(c\sqrt{n} - c\sqrt{\lambda T} > \epsilon\sqrt{n}) \\
&= \delta/3 + P\left(\frac{\sqrt{n}}{\sqrt{\lambda T}}(1-\epsilon/c) > 1\right) \\
&= \delta/3 + P\left(\frac{\sqrt{n}}{\sqrt{\lambda T}} > 1 + k\right) \\
&\leq \delta/3 + P\left(\frac{n}{\lambda T} > 1 + k^2\right) \\
&= \delta/3 + P\left(n - \lambda T > k^2 \lambda T\right) \\
&\leq 2\delta/3
\end{aligned}$$

for sufficiently large T, since $n - \lambda T \sim \sqrt{\lambda T}\, Z$, where Z is the standard normal.

A nearly identical argument shows that for large T,

$$P\left(\frac{d_T}{\sqrt{\lambda T}} - \frac{d_T}{\sqrt{n}} < -\epsilon\right) \leq \delta/3,$$

and the proof is complete.

REMARK 4.4. The relation between Theorem 4.1 and Corollary 4.2 is worth mentioning. The former result describes the behavior of the Poisson process normalized by the deterministic factor λT while the latter establishes its behavior when normalized by the random variable n. The expression in Corollary 4.2 is perhaps more relevant in applications, since typically λ is unknown.

The term *Poisson bridge* is typically used in reference to the Poisson process rescaled by a deterministic factor as in Theorem 4.1. In view of the similarities between B^0 and $\hat{M}$, the process $\hat{M}$ may be called a *stepping stone* process in analogy with the Brownian bridge process. Not only are the two processes related asymptotically by Corollary 4.2, but both display similar self-correcting behavior. Further, B^0 and $\hat{M}$ may be viewed as "tied down" versions of Brownian Motion and the Poisson process, respectively: $B^0(0) = B^0(1) = 0$; $\hat{M}_T(0) = \hat{M}_T(n) - n = 0$.

Corollary 4.2 suggests that the asymptotics of residual point processes may be described in relation to the Brownian Bridge. However the proof in Corollary 4.2 is given only for residuals of the Poisson process. The extension to more general point processes is given in the following conjecture.

CONJECTURE 4.5. Given certain restrictions on the parameterization of the conditional intensity of N, such as those in Ogata (1978), the result of Corollary 4.2 extends to the case where N is an arbitrary simple point process on the line.

5 Simulations of $\hat{M}$

The self-correcting behavior of $\hat{M}_T$ can be seen from simulations. Given the complexity of the conditional intensity of the process in (3.2), the simplicity with which one may simulate $\hat{M}_T$ is striking. The procedure is as follows:

- Generate n, a Poisson random variable with mean λT.

- Distribute n points uniformly on $[0, n]$.

The conditional intensity of $\hat{M}_T$ may also readily be simulated, using equation (3.2). Both the numerator and denominator in (3.2) generally converge rapidly for typical values of λ, T, m and t.

For all the simulations which follow, the product λT is chosen to be 10. This choice is arbitrary; however the results are similar for other relatively small values of λT.

Figure 1 shows ten simulations of $\hat{M}$; each row of points in Figure 1 represents one simulation. The regularity of the simulations in Figure 1 is of note: if $\hat{M}$ is observed from time 0 to 7, then $\hat{M}$ is guaranteed to have exactly 7 points in this interval.

The self-correcting behavior of $\hat{M}$ may be demonstrated graphically. Suppose we look at the processes in Figure 1 and focus on a certain interval $(t_1, t_2]$ of transformed time, e.g. $(4, 5]$. If $\hat{M}$ is indeed self-correcting, then we would expect to see relatively few points in $(4, 5]$ among processes that have many points in $[3, 4]$, and more points in $(4, 5]$ among processes with few points in $[3, 4]$.

Figure 2 shows how $\hat{M}(4, 5]$ relates to $\hat{M}[3, 4]$, for 1500 simulations. The data are perturbed slightly so that all the points can be seen. The dashed line is fit by least squares, the solid line by loess. The downward slope is readily apparent, confirming the self-correcting nature of $\hat{M}$.

The behavior of $\hat{M}$ can also be inspected by examining its conditional intensity $\lambda_{\hat{M}}(t)$ in (3.2). Figure 3 shows the conditional intensity process $\lambda_{\hat{M}}$ for the bottom-most simulated Poisson process shown in Figure 1. The points of $\hat{M}$ are depicted at the bottom of Figure 3. The volatility of $\lambda_{\hat{M}}$ is evident: note that if $\hat{M}$ were a unit-rate Poisson process, $\lambda_{\hat{M}}$ would be 1 everywhere. Instead, $\lambda_{\hat{M}}(t)$ ranges from less than .5 to more than 3.

From equation (3.2), given λT and t, the random variable $\lambda_{\hat{M}}(t)$ depends only on m, the number of points $\hat{M}$ has between 0 and t. In particular, if

$m < t$, then $\lambda_{\hat{M}}(t) > 1$, and if $m > t$, then $\lambda_{\hat{M}}(t) < 1$. Again, this verifies self-correcting behavior: when m is low (i.e. few points have occurred), $\lambda_{\hat{M}}$ is high, and vice versa.

Figure 4 shows how $\lambda_{\hat{M}}(t)$ decays with m, when $t = 4.5$. Although the general trend seen in Figure 4 appears to hold for various t, the rate of decay depends on t. When t is large, $\lambda_{\hat{M}}(t)$ decays very rapidly with m for m near t. This can be seen by comparing Figure 4 with Figure 5, which shows $\lambda_{\hat{M}}(t)$ as a function of m as in Figure 4, but with $t = 12.5$ instead of 4.5.

A perspective plot summarizing the general dependence of $\lambda_{\hat{M}}(t)$ on m and t is given in Figure 6. One sees that, for a given value of t, $\lambda_{\hat{M}}(t)$ decreases quickly as m exceeds t, and again that this decay is faster for larger t.

6 Summary and Conclusions

The rescaled or *stepping stone* process $\hat{M}$ investigated here appears to be a natural point process analog of the Brownian Bridge. Both processes are constrained at the bounds of their support, and they are closely related asymptotically as shown in Section 4. In contrast to the usual formulation of the Poisson bridge, the asymptotics investigated here are of a Poisson process rescaled according to the random number of observed points in an interval rather than a deterministic constant.

The process $\hat{M}$, arising from such simple and basic premises, is shown to have a very complex, self-correcting nature. This stems from the fact that $\hat{M}$ is guaranteed to average exactly one point per unit of transformed time. The situation is similar to the case of linear regression, where the residuals are guaranteed to have mean zero.

As demonstrated from both simulations and direct calculation, the self-correcting behavior in $\hat{M}$ is quite substantial. The conditional intensity of $\hat{M}$ is seen to vary wildly, rather than remain constant. The conclusion that $\hat{M}$ is essentially similar to a Poisson process therefore appears not to be justified.

The present work shows that even in the simplest case, where the original process N is a stationary Poisson process on the line, the residual process is far from Poisson when λT is small. Preliminary investigation suggests that the present results extend to the case where N is a more complicated point process, e.g. a non-stationary, non-Poissonian, and/or multi-dimensional point process.

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Figure Captions:

Figure 1: Ten simulations of $\hat{M}$

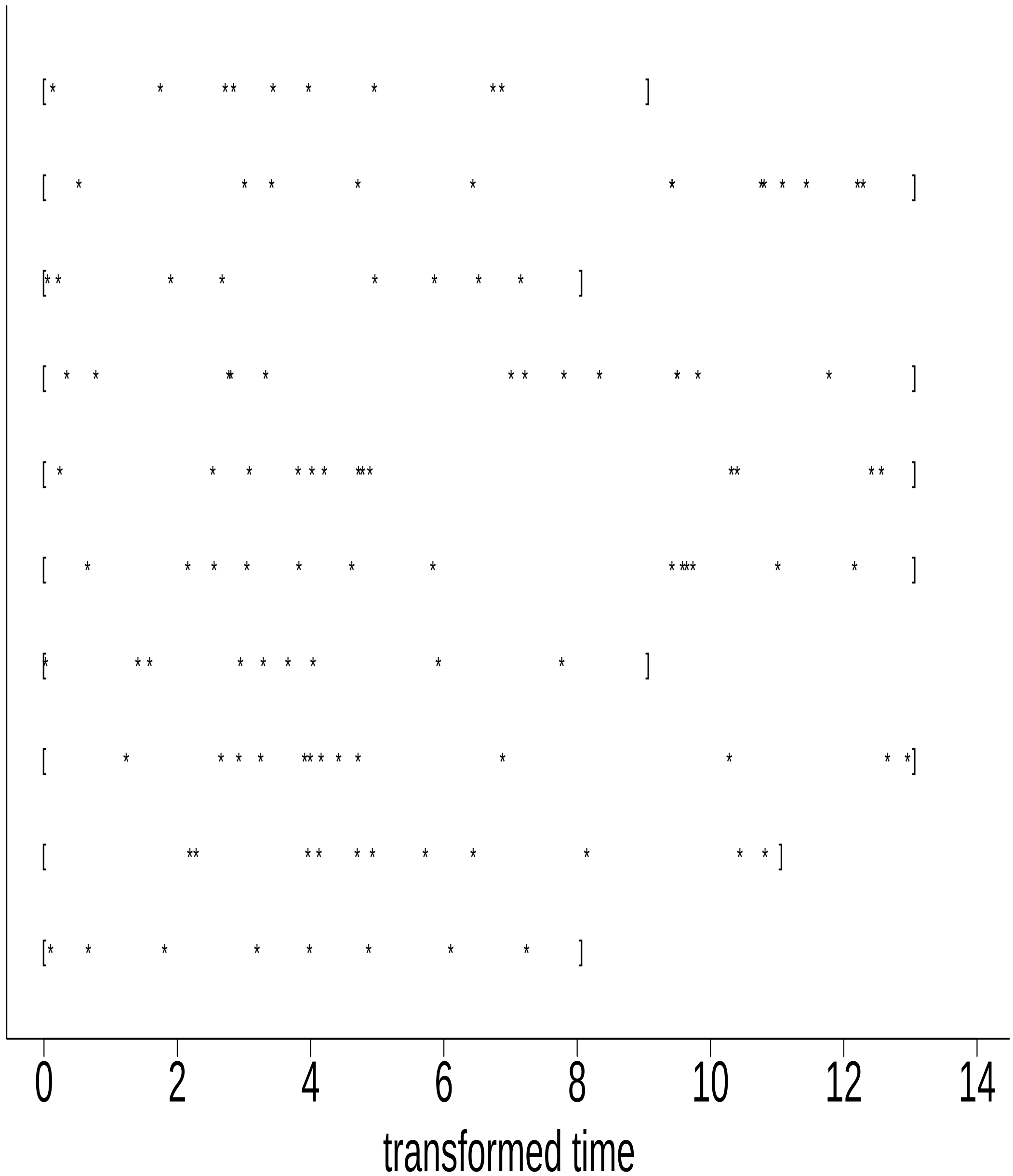
Figure 2: Plot of $\hat{M}(4,5]$ versus $\hat{M}[3,4]$, for 1500 simulations

Figure 3: Simulation of $\lambda_{\hat{M}}$

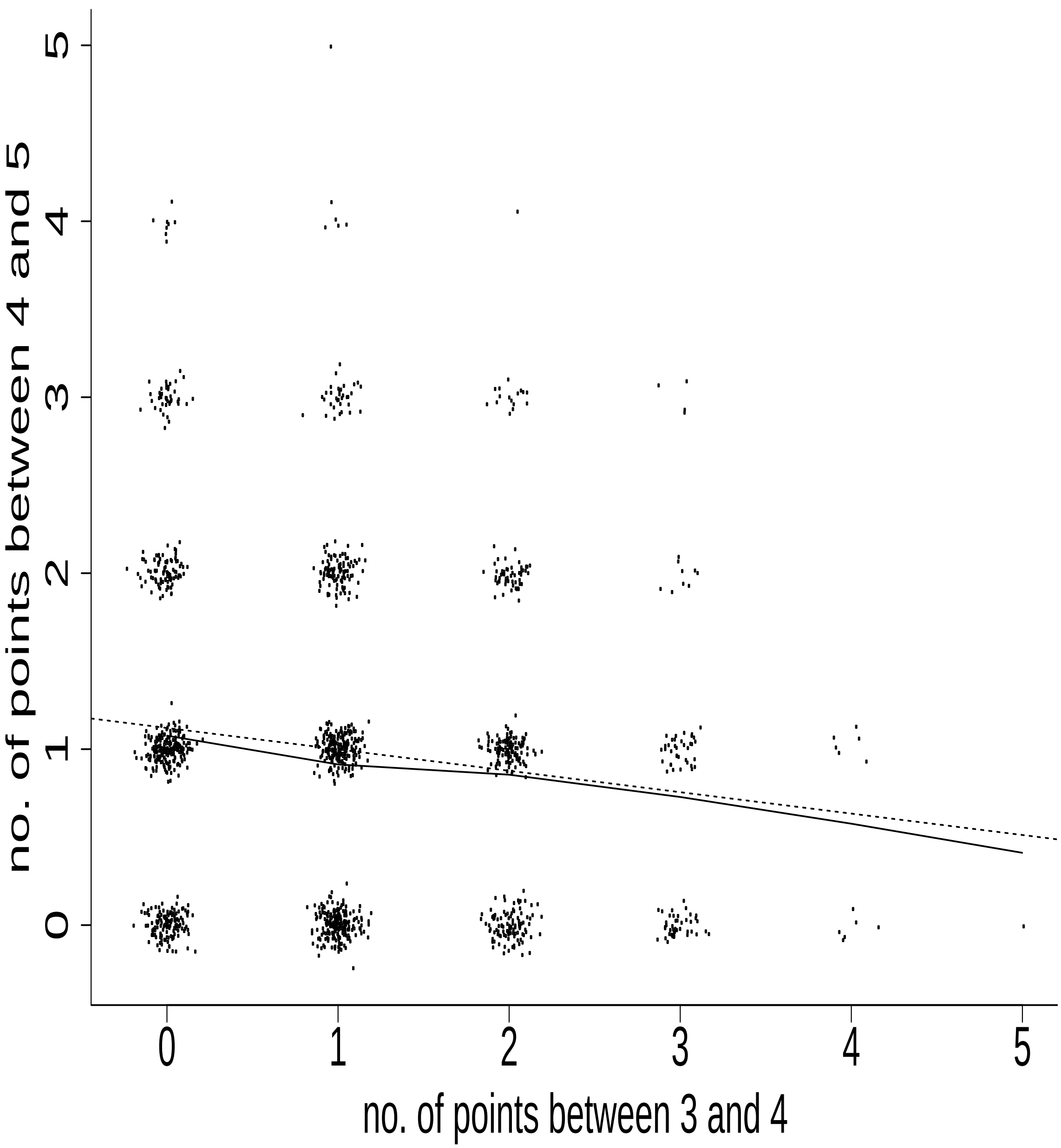
Figure 4: $\lambda_{\hat{M}}(t)$ vs. m, for $t = 4.5$

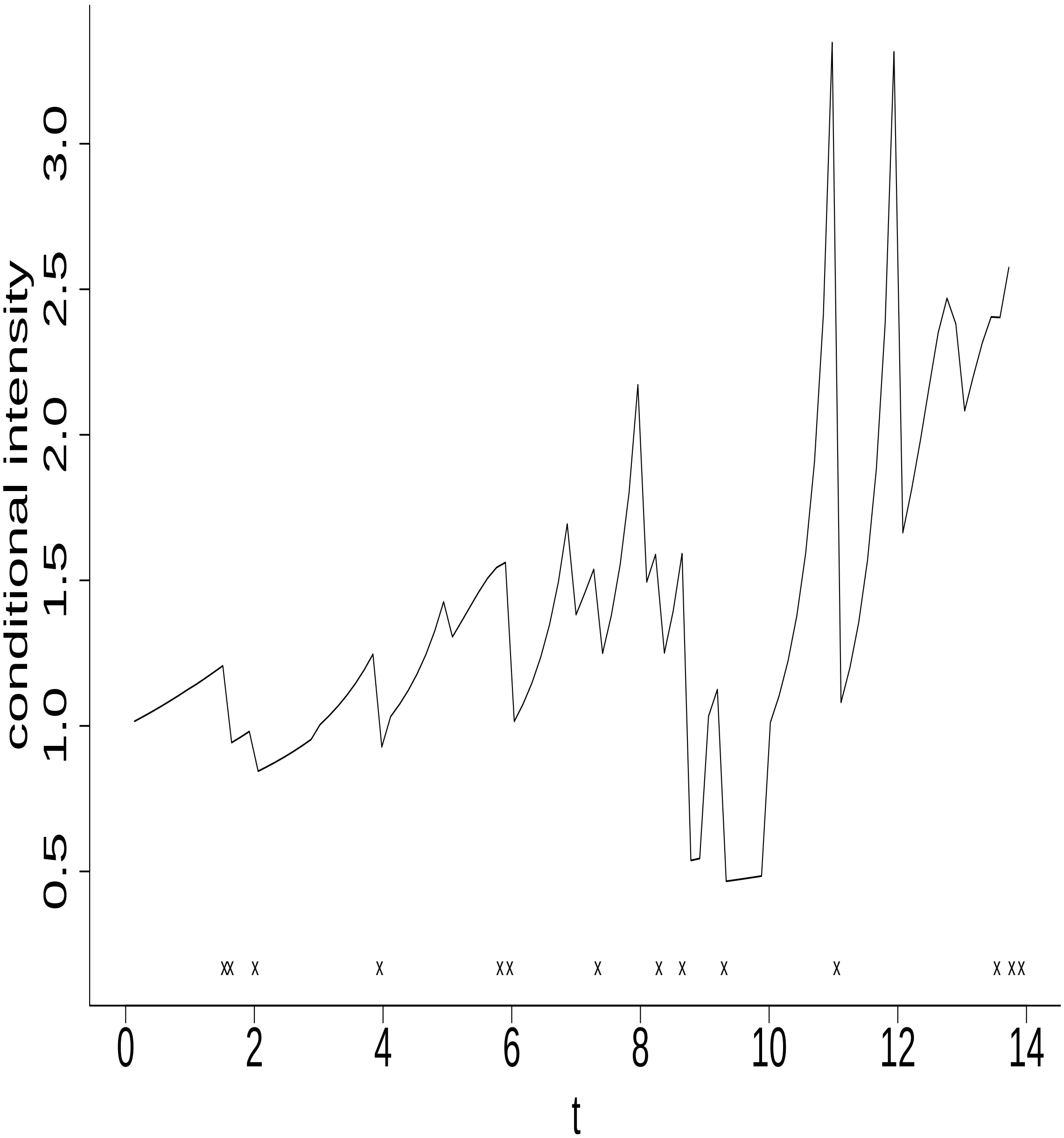
Figure 5: $\lambda_{\hat{M}}(t)$ vs. m, for $t = 12.5$

Figure 6: Perspective plot of $\lambda_{\hat{M}}(t)$ vs. t and m

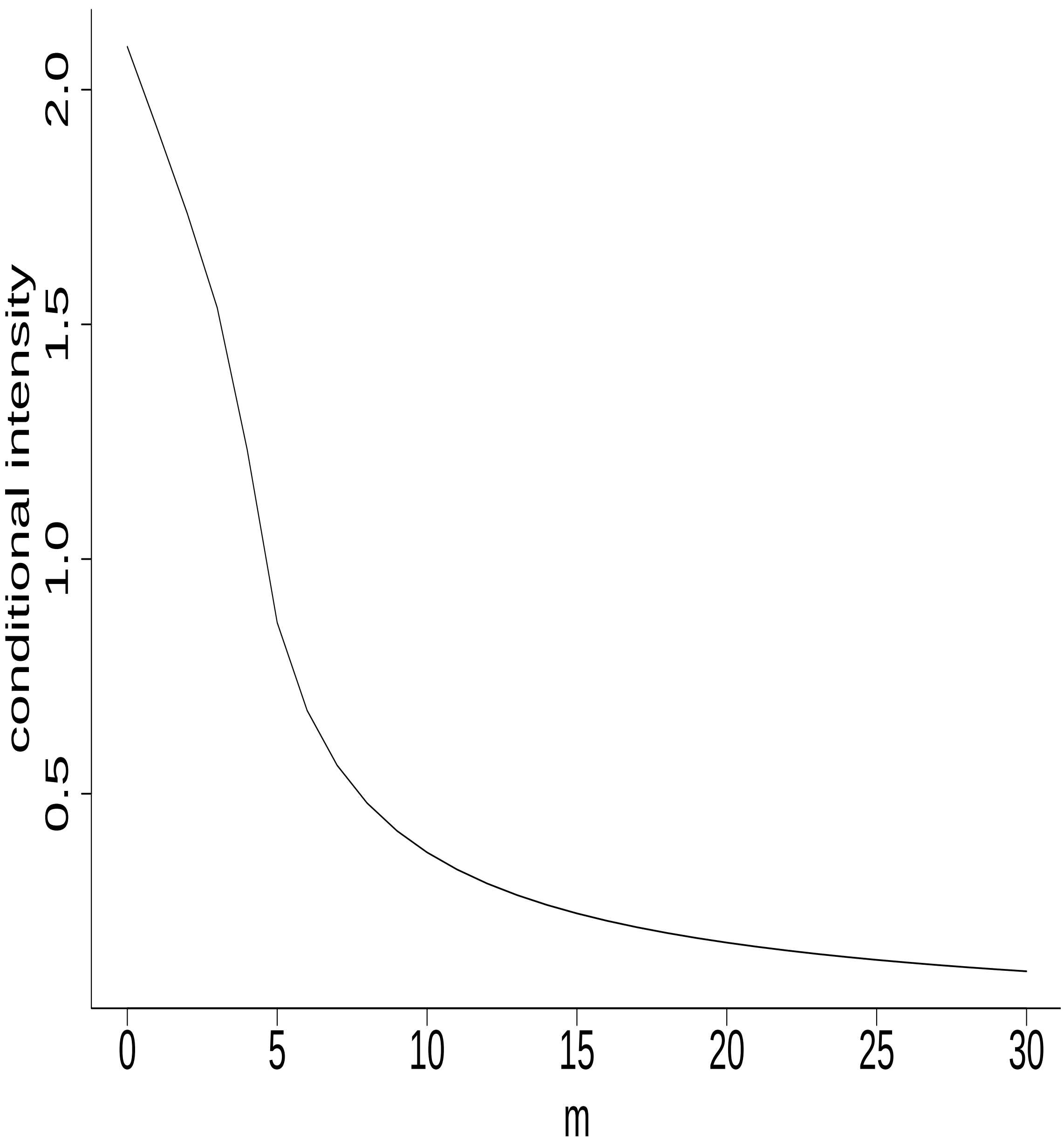


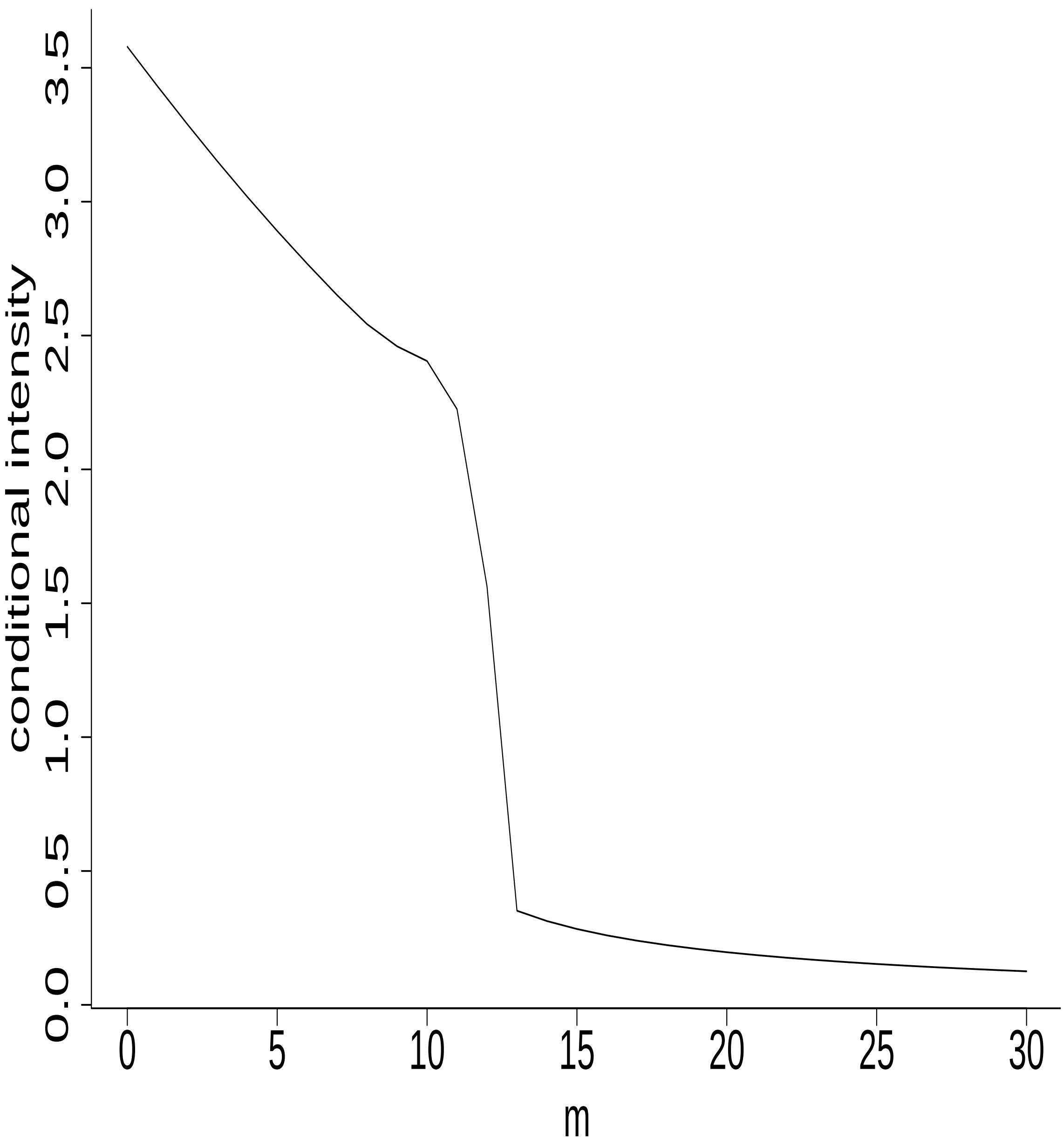
0
2
4
6
8
10
12
14
transformed time


no. of points between 4 and 5
0
1
2
3
4
5
0
1
2
3
4
5
no. of points between 3 and 4



conditional intensity
0.5
1.0
1.5
2.0
2.5
3.0
0
2
4
6
8
10
12
14
t


conditional intensity
2.0
1.5
1.0
0.5
0
5
10
15
20
25
30
m


conditional intensity
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0
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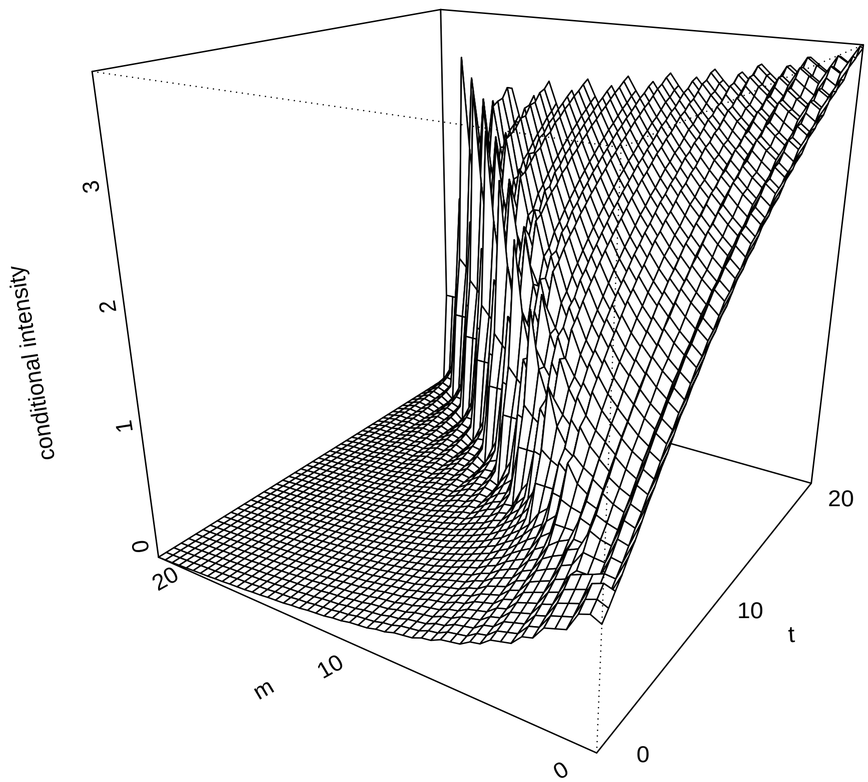
10
15
20
25
30
m


conditional intensity
0
1
2
3
20
m
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0
0
10
t
20