# Point processes, spatial

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The general theory of point processes has usually been focused on temporal developments (*see* **Point processes, temporal**). The special case of events that occur in space (or, more precisely, on a plane) has developed some particular approaches, usually assuming spatial *stationarity* and *isotropy* (i.e. the distribution of the process is invariant under translations and rotations, respectively). In this entry we will focus on this type of process. Points, or events, take place at spatial locations, and we identify each event with its location.

As an example, Figure 1 shows locations that experienced damage of modified Mercalli intensity (MMI) level VI in the 1989 Loma Prieta, CA, earthquake. MMI level VI is defined, in part, as

> Damage slight in specially designed structures: considerable in ordinary substantial buildings, with partial collapse; great in poorly built structures. Fall of chimneys, ...

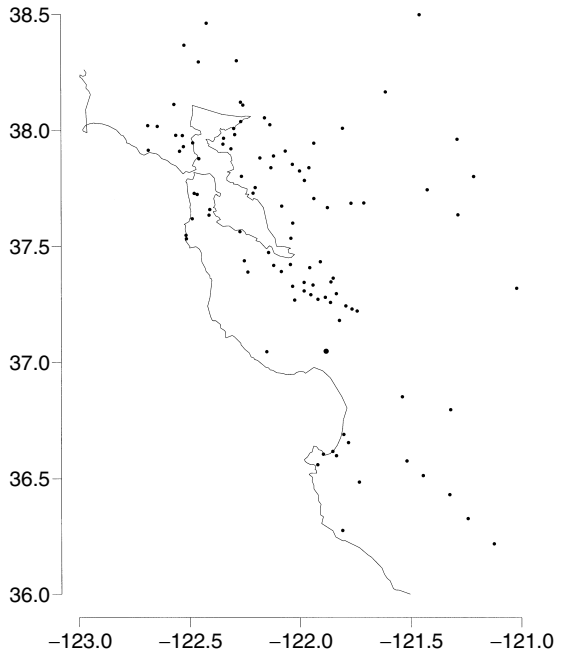


**Figure 1** Locations that experienced damage of MMI level VI in the 1989 Loma Prieta, CA, earthquake. The large dot denotes the epicenter of the quake

Much of the spatial point process theory originated with Matérn's work in **forestry** [6]. The book by Diggle [3] is a good summary of the state of affairs up to that time. Further aspects of point pattern analysis can be found in [2, Chapter 8] and in [14]. The latest developments have involved computationally intensive likelihood analysis [4].

## Second-order Parameters

The overall rate of a stationary, isotropic point process is measured by the parameter $\lambda_1 = \lim_{|A|\to 0} |A|^{-1}\mathrm{E}\left[\int_A \mathrm{d}N(x)\right]$, where $N(A)$ denotes the number of events in the (nice) set $A$ of Lebesgue measure $|A|$. A natural second-order parameter is the similarly defined parameter measure $\lambda_2(x) = \lim_{|A|\to 0} |A|^{-2}\mathrm{E}\left[\int_{y\in A} \mathrm{d}N(y)\,\mathrm{d}N(y+x)\right]$. This measures the chance of obtaining points simultaneously in the set $A$ and the set translated by a distance $x$.

A related parameter, proposed by Ripley [12] (*see* **Ripley's *K* function**), is $K(x) = \mathrm{E}$(number of points within $x$ of an arbitrary point). For an *orderly process*, $K$ is related to $\lambda_2$ by

$$K(x) = \frac{2\pi}{\lambda_1}\int_0^x u\lambda_2(u)\,\mathrm{d}u \tag{1}$$

i.e. a smoothed version of $\lambda_2$. An advantage of $K$ over $\lambda_2$ is that from a statistical point of view $K$ is easier to estimate, since it does not require choosing a smoothing parameter and is more stable. However, the intensity estimate has the advantage of making the fine detail clearer.

Another second-order parameter that can be estimated straightforwardly from data is the nearest neighbor distribution, i.e. the distribution of the distance from an arbitrary point to the location of its nearest neighbor point (*see* **Nearest neighbor methods**).

## Some Special Models

The simplest model for point processes is the *completely random*, or **Poisson process**. To define it, assume that there is a finite measure $\Lambda$, such that for all finite families of disjoint intervals $A_1, \ldots, A_n$

we have

$$\Pr[N(A_i = n_i,\ i = 1, \ldots, k)] = \prod_{i=1}^{k} \frac{\Lambda(A_i)^{n_i}}{n_i!} \exp[-\Lambda(A_i)] \tag{2}$$

In particular, the counts in disjoint sets are independent, and hence one cannot improve the prediction of the number of points in a set from information about numbers of points in, say, surrounding sets. This is what constitutes the complete randomness of the Poisson process. Often the Poisson process is taken as a null hypothesis to be rejected in favor of some more scientifically relevant process. Besag and Diggle [1] discuss how to assess such a pattern (as well as more complex ones) using Monte Carlo testing, which enables a researcher to test specific hypotheses by simulating the assumed process, and then to check whether the observed statistic of interest is extreme among the simulations (*see* **Simulation and Monte Carlo methods**).

The simplest type of Poisson process is isotropic with $\Lambda(A_i) = \lambda|A_i|$, where $\lambda_1 = \lambda$, $\lambda_2(x) = \lambda^2$ and $K(x) = \pi x^2$.

The most common deviation from complete randomness is clustering, i.e. a larger tendency for having points close to each other. A cluster point process is defined in a conditional two-step fashion. Given a point process $N_c$ of cluster centers, one associates with each of its events a secondary point process $N_s(\cdot|x)$, centered at an event at $x$. The cluster process is then the superposition of these secondary processes, or

$$N(A) = \int_x N_s(A|x) N_c(\mathrm{d}x) \tag{3}$$

A large class of cluster processes has a Poisson process of cluster centers. It is easy to see that such a **Poisson cluster process** is overdispersed with respect to a Poisson process with the same mean measure, i.e. that the cluster process has larger variance than the completely random process of the number of events in a set. The most common Poisson cluster process is the Neyman–Scott process, in which a random number of points are laid out in an independent and identically distributed fashion around the cluster centers. It was first introduced by Neyman [7] to describe the dispersion of larvae on a field, and has since found applications in astronomy [8, 10] and hydrology [5, 11]. The rate of a Neyman–Scott process is $\lambda_1 = \lambda_c \mathrm{E}(S)$, where $\lambda_c$ is the rate of the cluster center process, and $S$ is a random variable describing the number of points in a cluster. Also, $K(x) = \pi x^2 + [\mathrm{E}S(S-1)H_2(x)/\lambda_c \mathrm{E}^2 S]$, where $H_2$ is the cdf of the distance between two points in the same cluster.

A different class of point process models are the Markov point processes (*see* **Markov process**), introduced by Strauss [15] and by Ripley and Kelly [13]. A point process on a finite region $A$ is Markov with range $\delta$ if the conditional density of a point at location $x$, given the locations of all other points in $A$, depends only on the points within radius $\delta$ of $x$. Two points are called *neighbors* if they are within distance $\delta$ of each other. A set of mutual neighbors is called a *clique*. An advantage with the Markov point processes (shared with other processes defined through their conditional intensities) is that it is possible to write down a likelihood function for them (technically, this is a Radon–Nikodým derivative with respect to a unit rate Poisson process). It turns out [13] that a point process is Markov if and only if its likelihood depends only on the locations of cliques. A simple example of a Markov point process is the Strauss process. Let $s(x_1, \ldots, x_n)$ count the number of points within distance $\delta$ of each other. Then the *exponential family* with natural sufficient statistic $(n, s)$ and likelihood $L(\theta) = \exp[n\theta_1 + s(x_1, \ldots, x_n)\theta_2]/c(\theta)$, where $\theta = (\theta_1, \theta_2)$, is called the (unconditional) Strauss process. The process is defined for $\theta_2 \leq 0$, with $\theta_2 = 0$ corresponding to a Poisson process. When $\theta_2 < 0$ the resulting process is underdispersed relative to a Poisson process of the same rate $\theta_1$, so this process is sometimes used to describe patterns that are more regular than those from the Poisson process.

## Statistical Inference

Ripley [12] suggested estimating $K(x)$ from points $x_1, \ldots, x_n$ in a set $A$ by

$$\hat{K}(x) = \frac{|A|}{n^2} \sum_{i \neq j} 1 \frac{u_{ij<x}}{w_{ij}} \tag{4}$$

where $u_{ij} = |x_i - x_j|$ and $w_{ij}$ is the proportion of the area of a circle centered at $x_i$ with radius $u_{ij}$ that is inside $A$. This nonparametric estimator can be used to fit a parametric model by minimizing the distance

between the estimate and the parametric form of the function. A variety of examples can be found in [3].

Since the Strauss process (and other Markov point process models, as well as models developed using conditional intensities as in the entry on **point processes, spatial–temporal**) allows for a straightforward likelihood operation, apart, perhaps, from evaluating the normalizing constant $c(\theta)$, modern inference techniques, such as **Markov chain Monte Carlo** techniques, can be used to compute parameter estimates ([4] contains details and some worked examples; see also [9]).

## *References*


[1] Besag, J. & Diggle, P.J. (1977). Simple Monte Carlo tests for spatial pattern, *Applied Statistics* **26**, 327–333.

[2] Cressie, N. (1991). *Spatial Statistics*, Wiley, New York.

[3] Diggle, P.J. (1983). *Statistical Analysis of Spatial Point Patterns*, Academic Press, London.

[4] Geyer, C.J. (1999). Likelihood inference for spatial point processes, in *Stochastic Geometry: Likelihood and Computation*, W. Kendall, O. Barndorff-Nielsen & M.N.M. van Lieshout, eds, Chapman & Hall/CRC Press, London, pp. 141–172.

[5] Kavvas, M.L. & Delleur, J. (1981). A stochastic cluster model of daily rainfall occurrences, *Water Resources Research* **17**, 1151–1160.

[6] Matérn, B. (1960). *Spatial Variation*, Meddelanden från Statens Skogsforskningsinstitut No. 49.

[7] Neyman, J. (1939). On a new class of 'contagious' distributions, applicable in entomology and bacteriology, *The Annals of Mathematical Statistics* **10**, 35–37.

[8] Neyman, J. & Scott, E.L. (1959). Large scale organization of the distribution of galaxies, *Handbuch der Physik* **53**, 416–444.

[9] Ogata, Y. & Tanemura, M. (1984). Likelihood analysis and spatial point patterns, *Journal of the Royal Statistical Society* **46**, 496–518.

[10] Peebles, P.J.E. (1980). *The Large-scale Structure of the Universe*, Princeton University Press, Princeton.

[11] Phelan, M.J. & Goodall, C.R. (1990). An assessment of a generalized Waymire–Gupta–Rodriguez–Iturbe model for GARP Atlantic Tropical Experiment rainfall, *Journal of Geophysical Research* **95**, 7603–7616.

[12] Ripley, B.D. (1976). The second-order analysis of spatial point processes, *Journal of Applied Probability* **13**, 255–266.

[13] Ripley, B.D. & Kelly, F.P. (1977). Markov point processes, *Journal of the London Mathematical Society* **15**, 188–192.

[14] Stoyan, D., Kendall, W.S. & Mecke, J. (1987). *Stochastic Geometry and Its Applications*, Wiley, New York.

[15] Strauss, D.J. (1975). A model for clustering, *Biometrika* **63**, 467–473.


(*See also* **Point processes, dynamic**)

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