Point processes, temporal

David R. Brillinger, Peter M. Guttorp & Frederic Paik Schoenberg

Volume 3, pp 1577–1581

in

Encyclopedia of Environmetrics
(ISBN 0471 899976)

Edited by

Abdel H. El-Shaarawi and Walter W. Piegorsch

Point processes, temporal

A temporal point process is a random process whose realizations consist of the times $\{\tau_j\}$, $\tau_j \in \mathbb{R}$, $j = 0, \pm 1, \pm 2, \ldots$ of isolated events scattered in time. A point process is also known as a counting process or a random scatter. The times may correspond to events of several types.

Figure 1 presents an example of temporal point process data. The figure actually provides three different ways of representing the timing of floods on the Amazon River near Manaus, Brazil, during the period 1892–1992 (*see* **Hydrological extremes**) [7].

The formal use of the concept of point process has a long history going back at least to the life tables of Graunt [14]. Physicists contributed many ideas in the first half of the twentieth century; see, for example, [23]. The book by Daley and Vere-Jones [11] is an encyclopedic account of the theory of the subject.

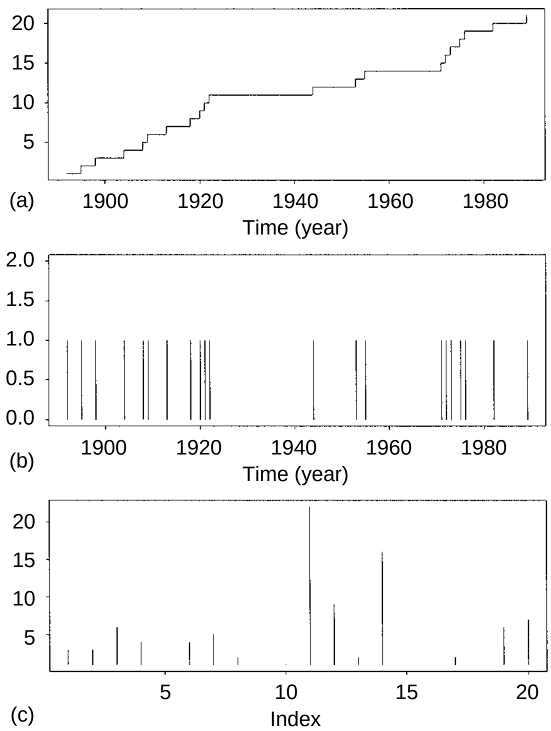

Figure 1 Floods on the Amazon River near Manaus, Brazil, during the years 1892–1992. (a) Amazon floods – cumulative count; (b) dates of floods; (c) intervals between floods

Examples

Examples of point processes abound in the environment; we have already mentioned times of floods. There are also times of earthquakes, fires, deaths, accidents, hurricanes, storms (hale, ice, thunder), volcanic eruptions, lightning strikes, tornadoes, power outages, chemical spills (*see* **Meteorological extremes**; **Natural disasters**).

Questions

The questions that scientists ask involving point process data include the following. Is a point process associated with another process? Is the association between two point processes actually causal? Is there a change or trend in time (*see* **Trend, detecting**)? Does the structure change (*see* **Change, detecting**)? Are the times clustered? Are the times repelled from each other? What is the predicted behavior? What is the risk (probability) of some event of negative consequence occurring at some future time (*see* **Risk assessment, probabilistic**)? How does one learn or describe the relationship of such processes? How does one carry out system identification?

Representations

A number of methods are used for the representation of point processes and of point process data. The figure shows three types of displays. The representations include:

- *step function*

$$N(t) = \#\{0 < \tau_j \le t\} \tag{1}$$

- *generalized function* (involving the Dirac delta function)

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \sum_j \delta(t - \tau_j) \tag{2}$$

- *counting measure*

$$N(I) = \#\{\tau_j \in I\} \tag{3}$$

- *binary time series*

$$dN(t) = \begin{cases} 1, & \text{if some point in } (t, t + dt] \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

(one might also write $N(dt)$ or $N(t, t + dt]$ here)

- *interevent intervals* $\{X_j\}$

$$X_j = \tau_{j+1} - \tau_j \quad (5)$$

assuming $\tau_{j+1} - \tau_j$ are on non-negative. In these expressions $\#\{A\}$ refers to the number of elements in the set A.

Distinctions

There are a variety of distinctions that may be made concerning types of point processes. A process may be either deterministic or stochastic. In the deterministic case the values τ_j are fixed. In the latter case (*see* **Stochastic process**) the process is determined by a consistent collection of probabilities such as

$$\Pr\{N(I_1) = n_1, \ldots, N(I_K) = n_K\}, \quad K = 1, 2, \ldots \quad (6)$$

where the I_k are Borel sets of the real line.

A process may be stationary, i.e. the time or space origin does not matter. A process may be mixing, i.e. distant values are only weakly related probabilistically. Points of the processes may be clumped together, i.e. clustered, or they may be repelled. In many cases a process is orderly, i.e. the points occurring are isolated.

Parameters

A variety of parameters provide useful descriptors of stochastic point processes. These include moments such as the rate

$$\frac{E\{dN(t)\}}{dt} \quad (7)$$

the auto-intensity

$$\frac{E\{dN(t + u)\, dN(t)\}}{dt\, du} \quad (8)$$

and the conditional rate

$$\frac{\Pr\{dN(t + u) = 1 | dN(t) = 1\}}{du} \quad (9)$$

functions. In the stationary case these functions will not depend on t. The first two generalize to product densities $p_K(\cdot)$, $K = 1, 2, \ldots$, giving the relative probabilities with which the points of interest are distributed at prespecified locations in time. Specifically

$$\Pr\{dN(t_1) = 1, \ldots, dN(t_K) = 1\} = p_K(t_1, \ldots, t_K)\, dt_1 \ldots dt_K \quad (10)$$

for the t_k distinct and $K = 1, 2, \ldots$.

Under weak conditions, including being orderly, a point process is characterized by its conditional or complete intensity function, $\mu(\cdot)$, as in

$$\Pr\{dN(t) = 1 | H_t\} = \mu(t | H_t)\, dt \quad (11)$$

where H_t is the history $H_t = \{\tau_j \le t\}$.

Another general way to define an (orderly) point process is via its zero probability function, that is

$$\phi(I) = \Pr\{N(I) = 0\} \quad \text{for bounded } I \quad (12)$$

Specific Point Processes

There are a number of important point processes that arise in both theory and practice.

The *renewal process* has the property that the intervals between successive points are independent and identically distributed positive random variables.

The ***Poisson process*** has a variety of definitions. One is that the conditional intensity function is constant. Another is that the counts $N(I_1), \ldots, N(I_K)$ of points in disjoint intervals I_k are independent Poisson variates with consistent expected values $K = 2, 3, \ldots$. A Poisson process is characterized by its rate function.

For the *doubly stochastic Poisson process* a non-negative random rate process in continuous time is first realized. Then a Poisson process with that rate function is generated.

For a *cluster process* there is a sequence of cluster centers $\{\sigma_j\}$, then further point processes $\{u_{jk}, k = 1, 2, \ldots\}$ are generated for each j. The cluster process then consists of the times $\{\sigma_j + u_{jk}\}$ (*see* **Poisson cluster process**).

The *Neyman–Scott* and *Bartlett–Lewis processes* are particular cases of the cluster process. In the former the u_{jk} are independent and identically distributed. In the latter the $\{u_{jk}\}$ are renewal processes having the σ_j as points of origin.

Operations on Point Processes

There is a calculus or algebra for manipulating point processes. This involves functions of realizations of basic processes. The operations may be applied by nature or by an analyst. One might consider, for example, a linear functional of a point process such as

$$\int \log \psi(t)\, \mathrm{d}N(t) \tag{13}$$

for some function $\psi(\cdot)$.

In the operation of *superposition* several processes are involved. In the superposed process the identity of each process is ignored and the times retained. If there are two processes M and N, then the superposed process is $M + N$ with the count of points in the set I given by $M(I) + N(I)$.

A point process may be *thinned*. In this operation points are deleted randomly.

Time substitutions are useful. What is involved is that a process M is converted to a process N by writing $N(t) = M[\Lambda(t)]$ for some nondecreasing, possibly random, function $\Lambda(\cdot)$. Through such a substitution, general processes may be derived from a homogeneous Poisson process.

Another operation is random translation. Here the points of the process are shifted

$$\{\tau_j\} \to \{\tau_j + \varepsilon_j\} \tag{14}$$

with the $\{\varepsilon_j\}$ taken as random.

There are *point process systems* where a point process input is carried into a point process output. The mechanism is typically stochastic; see [5]. Random translation as illustrated by (14) provides an example. Another example is provided by a model satisfying

$$\frac{\Pr\{\mathrm{d}N(t) = 1|M\}}{\mathrm{d}t} = \mu + \int a(t-u)\, \mathrm{d}M(u) \tag{15}$$

when the input point process is M. This provides a point process analog of the **linear model**.

The expression (13) is the basis of the probability generating functional. This is defined as

$$G[\psi] = \mathrm{E} \exp\left\{\int \log \psi(t)\, \mathrm{d}N(t)\right\} \tag{16}$$

and is a useful tool for developing properties of a process; see [22].

Inference

There is now a fairly extensive literature concerning inference for point processes. One may refer to the various books listed at the end of this entry.

In particular large sample properties of histogram type estimates of product densities are developed in [4]. **Nearest neighbor methods** are studied in [13]. In the case of the conditional cross rate function

$$\frac{\Pr\{\mathrm{d}N(t+u) = 1|\mathrm{d}M(t) = 1\}}{\mathrm{d}u} \tag{17}$$

a histogram type estimate is provided by

$$\frac{\#\{|\tau_k - \sigma_j - u| < b/2\}}{bM(T)} \tag{18}$$

where the terms $M(T)\,\sigma_j$ come from the process M, the τ_j come from N, and b is a binwidth parameter. This estimate may be computed exceedingly rapidly. Its distribution is approximately proportional to a Poisson when the point process (M, N) is stationary and mixing.

There are a variety of useful *statistical models*. One may mention the Hawkes process where the conditional intensity function is given by [16]

$$\mu(t|H_t) = \mu + \int_0^\infty a(t-u)\, \mathrm{d}N(u) \tag{19}$$

There are models containing explanatory variables. The latter are useful for dealing with nonstationary processes for example. One means of constructing them is by multiplying an elementary conditional intensity function by a function of some given functional form.

The likelihood may be set down given an expression for the conditional intensity function. The result is

$$L(\theta) = \prod_j \mu(\tau_j|\theta) \exp\left\{-\int_0^T \mu(t|\theta)\, \mathrm{d}t\right\} \tag{20}$$

where the available data values are the points $\{\tau_j\}$ observed in the time interval $(0, T]$ and where θ is an unknown parameter. Large sample properties of estimates obtained by maximizing $L(\theta)$ are developed in [12].

There are **residual** analyses; see [19] and [20].

There are limit theorems leading to useful approximations for the distributions of statistics; e.g. the superposition of many processes often leads to the

Poisson process. There are frequency domain analyses [6, 19].

There are point process analogs of many of the concepts of **time series**; see [6]. For example, expression (8) provides a point process analog of the autocovariance function of zero mean time series analysis. Surprisingly the point process case is often simpler. The parameters may have more basic interpretations, e.g. as probabilities. In various circumstances one can use programs developed for **binary data** to (approximately) analyze point process data.

Continuous time series can lead to a point process. For example, there is the series of times of crossing a given threshold; see [18]. One can also consider the process of times of extreme values (*see* **Extreme value analysis**).

Extensions

Aalen [1] (see also [2]) recognized the utility in **survival analysis** of considering a vector of point processes whose components had but one event. This idea has been developed extensively.

There are now a variety of extensions of the concept of point process as discussed above. The domain of the process may be $\mathbb{R}^p$ or more general, e.g. elements in a function space. There are marked point processes whose realizations have the form $\{\tau_j, M_j\}$. When the marks are real-valued the process may be represented as

$$\sum_j M_j \delta(t - \tau_j) \tag{21}$$

There are time series, $\{Y(t) = \int a(t-u)\,\mathrm{d}N(u)\}$, created by taking a point process N as a building block. There are graphical models in which the nodal variables are point processes [8]. There are hybrid processes, $\{Y(\tau_j)\}$, arising by sampling signals.

Some Books

A number of books have been written on the subject of point processes. These include: [2], [3], [9]–[11], [15], [17] and [21].

References

[1] Aalen, O.O. (1975). Statistical inference for a family of counting processes, Ph.D. Thesis, University of California, Berkeley.

[2] Anderson, P.K., Borgan, O., Gill, R.D. & Keiding, N. (1993). *Statistical Models Based on Counting Processes*, Springer-Verlag, New York.

[3] Brémaud, P. (1981). *Point Processes and Queues*, Springer-Verlag, New York.

[4] Brillinger, D.R. (1975). Statistical inference for stationary point processes, in *Stochastic Processes and Related Topics*, Vol. 1, M.L. Puri, ed., Academic Press, New York, pp. 55–99.

[5] Brillinger, D.R. (1975). The identification of point process systems, *The Annals of Probability* **3**, 909–929.

[6] Brillinger, D.R. (1978). Comparative aspects of the study of ordinary time series and of point processes, in *Developments in Statistics*, Vol. 1, P.R. Krishnaiah, ed., Academic Press, New York, pp. 33–133.

[7] Brillinger, D.R. (1995). Trend analysis: binary-valued and point process cases, *Stochastic Hydrology* **9**, 207–213.

[8] Brillinger, D.R. (1996). Remarks concerning graphical models of time series and point processes, *Brazilian Journal of Econometrics* **16**, 1–23.

[9] Cox, D.R. & Isham, V. (1980). *Point Processes*, Chapman & Hall, London.

[10] Cox, D.R. & Lewis, P.A.W. (1966). *The Statistical Analysis of Series of Events*, Methuen, London.

[11] Daley, D.J. & Vere-Jones, D. (1988). *An Introduction to the Theory of Point Processes*, Springer-Verlag, New York.

[12] Dzhaparidze, K. (1985). On asymptotic inference about intensity parameters of a counting process, *Proceedings of the International Statistical Institute* **45**, 1–15.

[13] Ellis, S.P. (1991). Density estimation for point processes, *Stochastic Processes and Their Applications* **39**, 345–358.

[14] Graunt, J. (1665). *London's Dreadful Visitation*, Cotes, London.

[15] Guttorp, P.M. (1995). *Stochastic Modelling of Scientific Data*, Chapman & Hall, London.

[16] Hawkes, A.G. (1972). Spectra of some mutually exciting point processes with associated variables, in *Stochastic Point Processes*, P.A.W. Lewis, ed., Wiley, New York, pp. 261–271.

[17] Karr, A. (1986). *Point Processes and Their Statistical Inference*, Marcel Dekker, New York.

[18] Leadbetter, M.R. (1972). Point processes generated by level crossings, in *Stochastic Point Processes*, P.A.W. Lewis, ed., Wiley, New York, pp. 436–467.

[19] Lewis, P.A.W. (1972). Recent results in the statistical analysis of univariate point processes, in *Stochastic Point Processes*, P.A.W. Lewis, ed., Wiley, New York, pp. 1–54.

[20] Ogatta, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes, *Journal of the American Statistical Association* **83**, 9–27.

[21] Snyder, D.L. & Miller, M.I. (1991). *Random Point Processes in Time and Space*, Wiley, New York.
[22] Westcott, M. (1972). The probability generating functional, *Journal of the Australian Mathematical Society* **14**, 448–466.
[23] Yvon, J. (1935). *La Théorie statistique des Fluids et l'Équation d'État*, Herman, Paris.

(*See also* **Point processes, dynamic**; **Point processes, spatial**; **Point processes, spatial–temporal**; **Waiting time**).

DAVID R. BRILLINGER, PETER M. GUTTORP & FREDERIC PAIK SCHOENBERG