

Discussion on “Of quantiles and expectiles: consistent scoring functions, Choquet representation and forecast rankings”

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We congratulate the authors for an inspiring piece of work. This motivates several intriguing and practically important research directions. First we want to point out an interesting connection between the Murphy diagram and “backtests” of Christoffersen (1998), which is a popular procedure for validating Value-at-Risk (VaR). We start from a simple simulation study that compares two VaR predictors (for $\tau = 0.01$) through the Murphy diagram. Let Y_t follow

$$Y_t = 0.9Y_{t-1} + \varepsilon_t, \quad t = 1, \dots, T = 500, \quad \varepsilon_t \sim N(0, 1), \text{ i.i.d.} \quad (1)$$

Two ways of predicting VaR (inspired by Christoffersen (1998)) given the knowledge of (1):

$$\widehat{VaR}_1^\tau(t) = 1\% \text{ quantile of } N\left(0, \frac{1}{1-0.9^2}\right) = -5.3370; \quad (2)$$

$$\widehat{VaR}_2^\tau(t) = 0.9Y_{t-1} + 1\% \text{ quantile of } N(0, 1). \quad (3)$$

To evaluate VaR estimation, overall coverage and error independence are checked through $I^\tau(t) := \mathbf{1}\{Y_t < \widehat{VaR}^\tau(t)\}$ with $\mathbf{1}(\cdot)$ being the indicator function, i.e., (i) $\mathbb{E}[I^\tau(t)] = \tau$ and (ii) $I^\tau(t)$ is independent of the sigma algebra generated by $(I_{t-1}^\tau, I_{t-2}^\tau, \dots)$. As checked by the tests proposed in Christoffersen (1998), (2) and (3) satisfy (i), while they differ in (ii). Interestingly, the elementary scoring function $S_{\tau, \theta}^Q(x, y)$ used in the Murphy diagram echoes with these two VaR standards, where

$$S_{\tau, \theta}^Q(x, y) = \underbrace{\{\mathbf{1}(y < x) - \tau\}}_{(A)} \underbrace{\{\mathbf{1}(x < \theta) - \mathbf{1}(y < \theta)\}}_{(B)}. \quad (4)$$

By replacing x and y in (4) by $\widehat{VaR}_1^\tau(t)$ ($\widehat{VaR}_2^\tau(t)$) and Y_t , we find that (A) measures “coverage”, i.e., (i), and that (B) measures the quality of x mimicking the dynamics of y , i.e., (ii).

Murphy diagram (Figure 1) demonstrates that $\widehat{VaR}_1^\tau(t)$ “dominates” $\widehat{VaR}_2^\tau(t)$. This dominance pattern can be linked with standards (i) and (ii).

1. $\theta \in [-5.337, -2]$: The large score deviation in this region is mainly due to (B) caused by the

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fixed value of $\widehat{VaR}_1^\tau(t)$: for most t , $\mathbf{1}(\widehat{VaR}_1^\tau(t) < \theta) = 1$, while $\mathbf{1}(Y_t < \theta) = 0$;

2. $\theta \in (-2, 6]$: The score deviation is much less in this region because of similar (A) (due to (i)) and (B) of two VaRs, while the latter to the fact that $\widehat{VaR}_1^\tau(t), \widehat{VaR}_2^\tau(t) < \theta$ for most t .

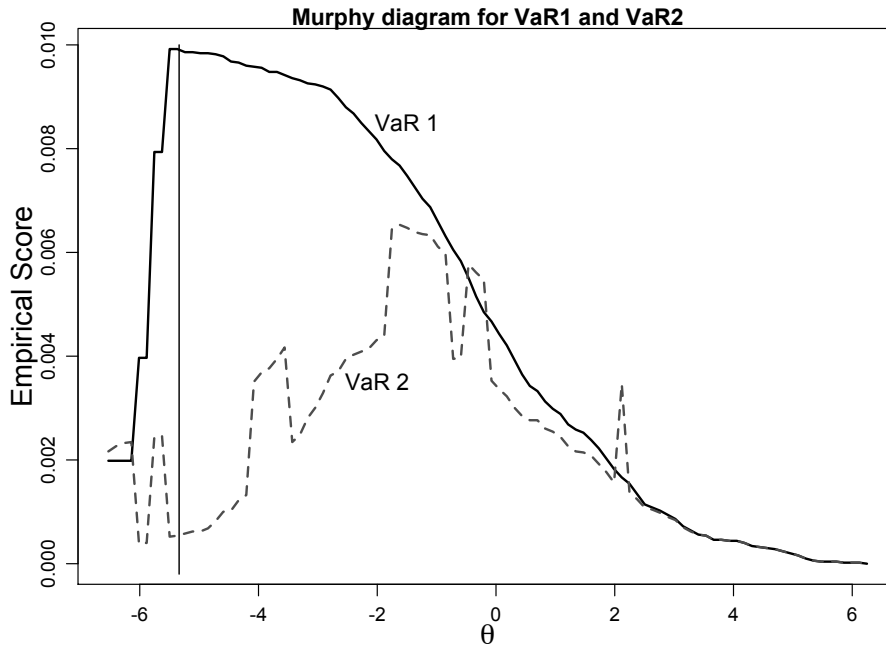


Figure 1: Murphy diagram $T^{-1} \sum_{t=1}^T S_{\tau,\theta}^Q(\widehat{VaR}_j^\tau(t), Y_t)$ with the score $S_{\tau,\theta}^Q(x, y)$ defined in (4): solid line for $\widehat{VaR}_1^\tau(t)$ and dashed line for $\widehat{VaR}_2^\tau(t)$. The vertical line corresponds to $\theta = -5.337$.

Another direction is on the detection of pattern change in dominance between two point forecasts. For example, the estimation quality for two VaRs in (2) and (3) may change when Y_t no longer follow an AR(1) model. Suppose $X_{1,t}$ and $X_{2,t}$ are two estimates for a certain quantile of Y_t , and $X_{1,t}$ dominates $X_{2,t}$ at $t = 0$. This is equivalent to $\text{sgn}(d_{t=0}(\theta)) = 1$ for all θ , where $d_t(\theta) = S_{\tau,\theta}^Q(X_{1,t}, Y_t) - S_{\tau,\theta}^Q(X_{2,t}, Y_t)$. To determine the critical point t_0 after which $X_{2,t}$ dominates $X_{1,t}$, we may consider a CUSUM statistics (Siegmund (1985)):

$$\tilde{S}_T(\theta) := \max_{0 \leq k \leq T} \{S_T(\theta) - S_k(\theta)\} \text{ where } S_T(\theta) = \sum_{t=1}^T \text{sgn}(d_t(\theta)). \quad (5)$$

We conjecture that the dominance relation changes if $\inf_{\theta} \tilde{S}_T(\theta) > b$ for some number b . Of course, an appropriate choice of (b, T) requires further study.

References

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