A Bayesian-inspired minimum aberration criterion for two-level multi-stratum factorial designs

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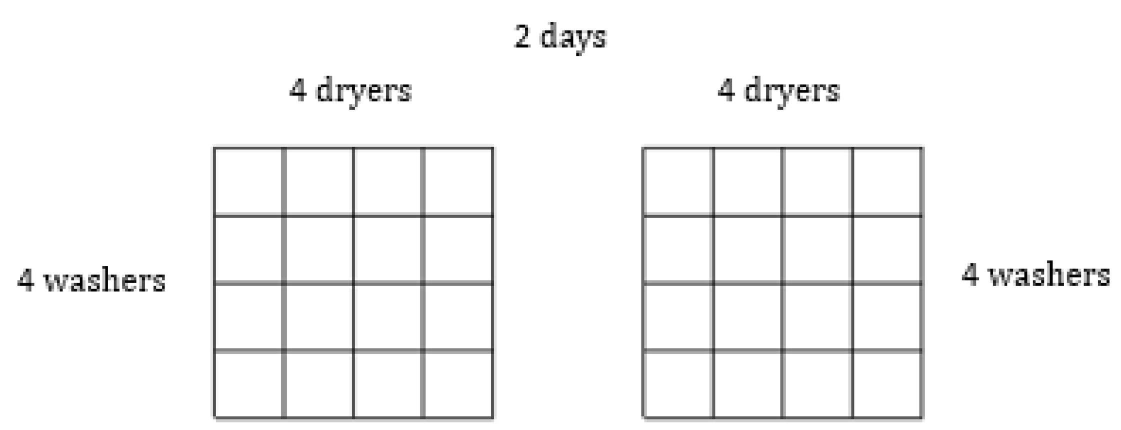
1. Introduction

In a multi-stratum factorial experiment, there are multiple error terms (strata) with different variances that arise from complicated structures of the experimental units. However, most existing criteria in the literature for selecting multi-stratum factorial designs are limited to certain scenarios. Mitchell, Morris, and Ylvisaker (1995) proposed a framework for two-level Bayesian factorial designs. We adopt this approach and propose a Bayesian-inspired minimum aberration criterion for selecting two-level multi-stratum factorial designs. A numerical example is used to illustrate the theory.

2. Multi-stratum experiments

The structure of experimental units, called ***block structure***, leads to multiple error terms (strata) with different variances. For example, block experiments, split-plot experiments, and strip-plot experiments are multi-stratum experiments. A laundry experiment in Miller (1997) is an example of blocked strip-plot experiments.



3. Literature review

Most existing minimum aberration criteria for selecting multi-stratum designs are modifications of the wordlength pattern proposed in Fries and Hunter (1980). Also, most of them are developed **ONLY** for *certain block structures* and/or *orthogonal regular factorial designs*. The following are three minimum aberration criteria for selecting orthogonal blocked regular factorial designs, from Sitter, Chen, and Feder (1997), Chen and Cheng (1999), Cheng and Wu (2002), respectively:

- $(A_{3,0}, A_{2,1}, A_{4,0}, A_{3,1}, A_{5,0}, \boldsymbol{A_{4,1}}, \boldsymbol{A_{6,0}}, \ldots)$
- $(A_{3,0}, A_{2,1}, A_{4,0}, A_{5,0}, \boldsymbol{A_{3,1}}, \boldsymbol{A_{6,0}}, A_{7,0}, \ldots)$
- $(A_{3,0}, A_{4,0}, A_{2,1}, A_{5,0}, \boldsymbol{A_{6,0}}, \boldsymbol{A_{3,1}}, A_{7,0}, \ldots)$

4. Model

Consider a fractional factorial design with n two-level treatment factors and the N experimental units having a block structure $\mathfrak{B} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_m\}$. The N units are *unstructured* if $m = 1$. We write $\mathcal{F}_i \prec \mathcal{F}_j$ if $\mathcal{F}_i$ forms a finer partition than $\mathcal{F}_j$ on the N units. Suppose

$$\mathbf{y} = \mathbf{U}\boldsymbol{\beta} + \sum_{i=0}^{m} \mathbf{X}_{\mathcal{F}_i} \boldsymbol{\gamma}^{\mathcal{F}_i},$$

where $\boldsymbol{\beta}$ consists of 2^n *factorial effects* β_S's, $S \subseteq \{1, \ldots, n\}$, following a prior induced by a Gaussian process, and $\boldsymbol{\gamma}^{\mathcal{F}_i}$ consists of the *random unit effects* of $\mathcal{F}_i$. If $\mathfrak{B}$ satisfies some mild conditions, e.g., $\mathfrak{B}$ is an *orthogonal block structure*, $\mathrm{cov}(\mathbf{y}|\boldsymbol{\beta})$ has $m+1$ eigenvalue-eigenspace pairs $(\xi_{\mathcal{F}_i}, W_{\mathcal{F}_i})$, $i = 0, \ldots, m$. Each $W_{\mathcal{F}_i}$ is a ***stratum*** and $\xi_{\mathcal{F}_i}$ is a ***stratum variance***. Define

$$B_{k,i} = \frac{1}{N} \sum_{S:|S|=k} \|\mathbf{P}_{W_{\mathcal{F}_i}} \mathbf{u}_S\|^2$$

for $k = 0, \ldots, n$ and $i = 0, \ldots, m$, where $\mathbf{P}_{W_{\mathcal{F}_i}}$ is the orthogonal projection matrix onto $W_{\mathcal{F}_i}$.

5. A new minimum aberration criterion

Chang and Cheng (2017+) showed that sequentially minimizing the following pattern $W(d)$ is a good surrogate for **minimizing** $\mathbf{det}[\mathbf{cov}(\boldsymbol{\beta}|\mathbf{y})]$ under an effect-hierarchy prior:

$$W(d) = \left(\sum_{i=0}^{m-1} \left(\frac{1}{\xi_{\mathcal{F}_m}} - \frac{1}{\xi_{\mathcal{F}_i}} \right) B_{1,i}(d), \sum_{i=0}^{m-1} \left(\frac{1}{\xi_{\mathcal{F}_m}} - \frac{1}{\xi_{\mathcal{F}_i}} \right) B_{2,i}(d), \ldots, \sum_{i=0}^{m-1} \left(\frac{1}{\xi_{\mathcal{F}_m}} - \frac{1}{\xi_{\mathcal{F}_i}} \right) B_{n,i}(d) \right).$$

The following result provides a way to eliminate inferior designs **WITHOUT** knowing stratum variances.

Theorem 1. *A necessary and sufficient condition for a design to have minimum aberration with respect to $W(d)$ for **ALL feasible $\boldsymbol{\xi}$** is that it has minimum aberration with respect to*

$$W_{\mathfrak{G}}(d) = \left(\sum_{i:\mathcal{F}_i \in \mathfrak{G}} B_{1,i}(d), \sum_{i:\mathcal{F}_i \in \mathfrak{G}} B_{2,i}(d), \ldots, \sum_{i:\mathcal{F}_i \in \mathfrak{G}} B_{n,i}(d) \right)$$

for all subsets $\mathfrak{G}$ of $\mathfrak{B} \setminus \{\mathcal{F}_m\}$ satisfying

$$\mathcal{F} \in \mathfrak{G}, \mathcal{F}' \in \mathfrak{B}, \text{ and } \mathcal{F} \prec \mathcal{F}' \Rightarrow \mathcal{F}' \in \mathfrak{G}. \qquad (\bigstar)$$

6. Example

Consider 16-run blocked regular/nonregular factorial designs with four blocks of size four for five two-level treatment factors. Then $\mathfrak{B} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$. The subsets $\mathfrak{G}$ of $\mathfrak{B} \setminus \{\mathcal{F}_2\}$ satisfying $(\bigstar)$ are $\mathfrak{G}_1 = \{\mathcal{F}_0\}$ and $\mathfrak{G}_2 = \{\mathcal{F}_0, \mathcal{F}_1\}$. Let design d_1 be the projection of the 16-run Hall's design of Type II onto factors $\mathbf{1}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{t_{10}}, \mathbf{t_{12}}, \mathbf{t_{15}}$, with $\mathbf{4}$ and $\mathbf{6}$ as the block generators. Let design d_2 be be the projection of the same 16-run Hall's design onto factors $\mathbf{1}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{t_{10}}, \mathbf{t_{12}}, \mathbf{t_{15}}$, with $\mathbf{t_{12}}$ and $\mathbf{t_{15}}$ as the block generators. A complete search screens out all the designs except d_1 and d_2. Design d_1 has

$$W_{\mathfrak{G}_1}(d_1) = (B_{1,0}(d_1), \ldots, B_{5,0}(d_1)) = (0, 0, 0, 0, 1),$$
$$W_{\mathfrak{G}_2}(d_1) = (B_{1,0}(d_1) + B_{1,1}(d_1), \ldots, B_{5,0}(d_1) + B_{5,1}(d_1)) = (0, 3, 3, 0, 1),$$

and is optimal for $\mathfrak{G}_1$ ($\xi_{\mathcal{F}_1} = \xi_{\mathcal{F}_2}$), i.e., the case of unstructured units (as well as when the interblock variance $\xi_{\mathcal{F}_1}$ is not too larger than the intrablock variance $\xi_{\mathcal{F}_2}$). In contrast, d_2 has

$$W_{\mathfrak{G}_1}(d_2) = (B_{1,0}(d_2), \ldots, B_{5,0}(d_2)) = (0, 0, 0, 1, 0),$$
$$W_{\mathfrak{G}_2}(d_2) = (B_{1,0}(d_2) + B_{1,1}(d_2), \ldots, B_{5,0}(d_2) + B_{5,1}(d_2)) = (0, 2, 4, 1, 0),$$

and is optimal for $\mathfrak{G}_2$ ($\xi_{\mathcal{F}_0} = \xi_{\mathcal{F}_1} = \infty$), i.e., the case of fixed block effects (as well as when the interblock variance $\xi_{\mathcal{F}_1}$ is sufficiently greater than the intrablock variance $\xi_{\mathcal{F}_2}$). We suggest that d_2 be used under fixed block effects.

7. Conclusions

A Bayesian-inspired minimum aberration criterion is proposed for selecting two-level multi-stratum factorial designs. It is *statically meaningful* ($\leftrightarrow$ minimize $\det[\mathrm{cov}(\boldsymbol{\beta}|\mathbf{y})]$), applicable to

1. most block structures encountered in practice, e.g., blocked split-plot/strip-plot designs;
2. regular as well as nonregular/nonorthogonal factorial designs,

and need not assume that three- and higher-order interactions are negligible. A useful result (Theorem 1) can eliminate inferior designs when the stratum variances are unknown.

MAIN REFERENCES

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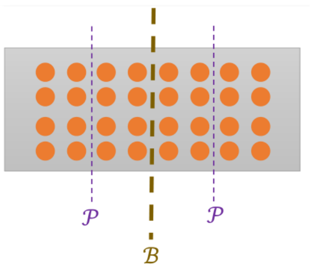
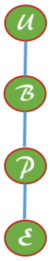
[2] Cheng, C. S. *Theory of Factorial Design: Single- and Multi-Stratum Experiments.* CRC Press, 2014.

[3] Mitchell, T. J., Morris, M. D., and Ylvisaker, D. Two-level fractional factorials and Bayesian prediction. *Statist. Sinica 5*, 2 (1995), 559–573.

Nesting

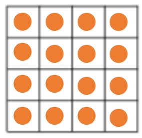


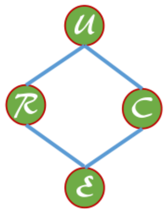


$$\mathfrak{B} = \{\mathcal{U}, \mathcal{B}, \mathcal{P}, \mathcal{E}\}$$

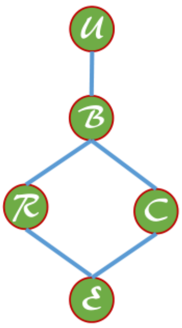
$$\mathcal{E} \prec \mathcal{P} \prec \mathcal{B} \prec \mathcal{U}$$

Crossing

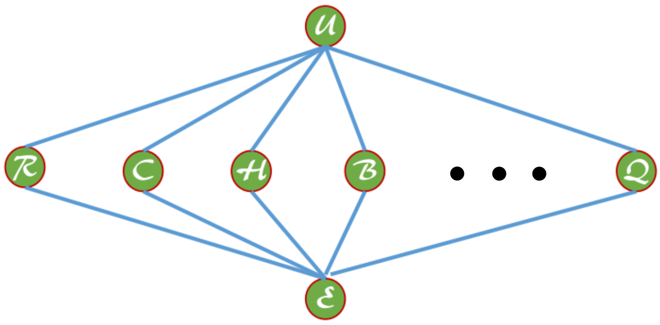




Simple block structure



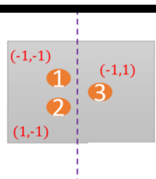
Orthogonal block structure



Linear mixed effect model

$N = 3, m = 2, n = 2, \mathfrak{B} = \{\boxed{\mathcal{F}_0}, \boxed{\mathcal{F}_1}, \boxed{\mathcal{F}_2}\}$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \beta_\phi \\ \beta_{\{1\}} \\ \beta_{\{2\}} \\ \beta_{\{1,2\}} \end{pmatrix} + \boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \gamma^{\mathcal{F}_0}} + \boxed{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1^{\mathcal{F}_1} \\ \gamma_2^{\mathcal{F}_1} \end{pmatrix}} + \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1^{\mathcal{F}_2} \\ \gamma_2^{\mathcal{F}_2} \\ \gamma_3^{\mathcal{F}_2} \end{pmatrix}}$$



Condition (★)

$\mathfrak{G} = \{\mathcal{U}\}$, unstructured units

$\{\mathcal{U}, \mathcal{B}\}$, fixed block effects

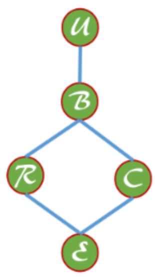
$\cancel{\{\mathcal{B}\}}$

$\mathfrak{G} = \{\mathcal{U}\}$, unstructured units

$\{\mathcal{U}, \mathcal{B}\}$, fixed block effects

$\{\mathcal{U}, \mathcal{B}, \mathcal{P}\}$ fixed block, whole-plot effects

$\cancel{\{\mathcal{U}, \mathcal{P}\}}$



$\mathfrak{G} = \{\mathcal{U}\}$, unstructured units

$\{\mathcal{U}, \mathcal{B}\}$, fixed block effects

$\{\mathcal{U}, \mathcal{B}, \mathcal{R}\}$, fixed block, row effects

$\{\mathcal{U}, \mathcal{B}, \mathcal{C}\}$, fixed block, column effects

$\{\mathcal{U}, \mathcal{B}, \mathcal{R}, \mathcal{C}\}$ fixed block, row, column effects

$\cancel{\{\mathcal{B}, \mathcal{R}, \mathcal{C}\}}$