A Class of Nonregular Designs Constructed from Quaternary Codes
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Abstract
The research of developing a general methodology for the construction of optimal nonregular design has been very active in the last 10 years. Recent research by Xu and Wong (2007) suggests a new class of nonregular designs constructed from quaternary codes. This paper explores the properties and uses of quaternary codes towards the construction of $2^{2(n-1)}$ runs nonregular designs. These designs are optimal under the maximum generalized resolution criterion over all quaternary-code designs and they are optimal under the generalized minimum aberration and maximum projectivity criteria over all possible designs. Some theoretical results are obtained and optimal designs up to $4^{10}$ runs are listed.

1. Introduction
In many scientific researches and investigations, the interests lie in the study of effects of many factors simultaneously. Factorial designs, especially two-level factorial designs, are the most commonly used experimental plans for this type of investigation. A full factorial experiment allows all factorial effects to be estimated independently. However, it is often too costly to perform a full factorial experiment, so a fractional factorial design, which is a subset or fraction of a full factorial design, is preferred since it is cost-effective. The price of this effectiveness is the aliasing of factorial effects so that some effects cannot be estimated.

Some fractional factorial designs can be constructed through defining relations among factors. They are called regular designs. Any two factorial effects of regular designs are either orthogonal or fully aliased with each other. All other designs that do not possess this kind of defining relationship are called nonregular designs. Factorial effects are partially aliased and these designs are used for run size economy and flexibility.

Regular designs are chosen by the maximum resolution criterion (Box and Hunter 1961) and its refinement, the minimum aberration criterion (Fries and Hunter 1980). The concepts of resolution and aberration for regular designs have been extended to nonregular designs. Deng and Tang (1999) proposed generalized resolution and minimum aberration.

For nonregular designs, Butler (2003b, 2004) developed some theoretic results and constructed some special generalized minimum aberration designs over all possible designs without computer search. Xu (2005) constructed several nonregular designs with 64, 128 and 256 runs and 7-16 factors from the Nordstrom and Robinson (1967) code, which is a well-known nonlinear code in coding theory.

The methodology of quaternary code designs was first proposed by Xu and Wong (2007).
They described a general method for constructing two-level nonregular designs using quaternary codes, and proposed a systematic construction procedure for \( 4^k \times (4^k - 2^k) \) designs and its double with resolution 3.5 for any \( k \). The corresponding regular designs have maximum resolution. They also presented a collection of nonregular designs with 16, 32, 64, 128 and 256 runs and up to 64 factors. The advantages of using quaternary codes to construct nonregular designs are its relatively straightforward construction procedure and simple design presentation. Since the designs are constructed by linear codes over \( \mathbb{Z}_4 \), we may use column indexes to describe these designs. More importantly, many nonregular designs constructed by quaternary codes have better statistical properties than regular designs of the same size in terms of resolution, aberration and projectivity.

This paper further considers the construction of two-level nonregular designs from quaternary codes. These designs are optimal under maximum generalized resolution, generalized minimum aberration and maximum projectivity criteria. In order to come up with the new construction methods, some new definitions and technical lemmas are presented and explored extensively. Section 2 reviews some definitions and notations in the areas of nonregular designs, design criteria and quaternary codes. Section 3 presents the main results of the paper, including four theorems and a optimal design table. These theorems bypass the tedious calculations of \( J \)-characteristics and characterize the designs directly from their generator matrices. Then these theorems suggest the optimal design construction method simply from the structure of the generator matrix. In section 4, the main results are proved using some additional definitions and technical lemmas.

2. Definitions and Notations

2.1 \( J \)-characteristics and Aliasing Index

A design \( D \) of \( n \) runs and \( m \) factors is represented by an \( n \times m \) matrix, where each row corresponds to a run and each column to a factor. A two-level design takes on only two symbols, say \(-1\) and \(+1\). For \( s = \{c_1, c_2, \ldots, c_k\} \), a subset of \( k \) columns of \( D \), define \( j_k(s) = \sum_{i=1}^{k} c_{il} \), where \( c_{il} \) is the \( i \)-th entry of column \( c_l \). The \( J_k(s) = |j_k(s)| \) values are called the \( \textit{J-characteristics} \) of design \( D \) (Deng and Tang 1999). When \( D \) is a regular design, \( j_k(s) \) takes on only two values: 0 or \( N \). In general, \( 0 \leq j_k(s) \leq 1 \). If \( j_k(s) = n \), these \( k \) columns in \( s \) form a word of length \( k \).

According to the bounds of \( J \)-characteristics from 0 to \( N \) in non-regular design, we define a quantity called \( \textit{aliasing index} \) (\( \rho \)) to be the ratio between the \( J \)-characteristic of a subset \( s = \{c_{l1}, c_{l2}, \ldots, c_{lk}\} \) and the run size \( n \), i.e.

\[
\rho_k(s) = \frac{j_k(s)}{n}
\]

This quantity can be interpreted as a measure of aliasing of the columns in \( s \). It is
obvious that $0 \leq \rho_k(s) \leq 1$. When $\rho_k(s) = 0$, the columns in $s$ are not aliased. When $\rho_k(s) = 1$, the columns in $s$ are fully aliased with each other and form a **complete word** of length $k$. When $0 < \rho_k(s) < 1$, the columns in $s$ are partially aliased with each other and form a **partial word** of length $k$ with aliasing index $\rho_k(s)$.

### 2.2 Resolution, Aberration and Projectivity

Suppose that $r$ is the smallest integer such that $\max_{|s|=r} J_r(s) > 0$, where the maximization is over all subsets of $r$ columns of $D$. The **generalized resolution** of $D$ (Deng and Tang 1999) is defined as

$$GR(D) = r + \left(1 - \max_{|s|=r} \rho_r(s)\right) = r + \left(1 - \frac{\max_{|s|=r} J_r(s)}{n}\right)$$

Let

$$A_k(D) = \sum_{|s|=k} (\rho_k(s))^2 = \frac{1}{n^2} \sum_{|s|=k} (J_k(s))^2$$

which measures the number of words of length $k$. The vector $A_1(D), A_2(D), \ldots, A_m(D)$ is called the **generalized wordlength pattern** (GWLP) (Deng and Tang 1999). The **generalized minimum aberration criterion**, also called minimum $G_2$-aberration, is to sequentially minimize the components in the wordlength pattern $A_1(D), A_2(D), \ldots, A_m(D)$. This means if two designs have $A_k(D)$ to be the first non-zero component in the wordlength pattern, a design with lower $A_k(D)$ is preferred. A design $D$ is said to be of **projectivity** $p$ if every subset of $p$ factors out of the possible $n$ contains a complete $2^p$ factorial design, possibly with some points replicated (Box and Tyssedal 1996).

When the designs are restricted to regular designs, generalized resolution, generalized wordlength pattern and generalized minimum aberration reduce to the traditional resolution (Box and Hunter 1961), wordlength pattern and minimum aberration (Fries and Hunter 1980), respectively. In addition, according to Box and Hunter (1961), a regular design of resolution $R$ has projectivity $p = R - 1$.

### 2.3 Quaternary Linear Codes

A quaternary code takes on values from $\mathbb{Z}_4 = \{0, 1, 2, 3\} \pmod{4}$. Let $G$ be an $N \times l$ generator matrix over $\mathbb{Z}_4$. All possible linear combinations of the rows in $G$ over $\mathbb{Z}_4$ form a quaternary linear code, denoted by $C$. In order to obtain a two-level design, we apply the Gray map:

$$\phi: 0 \rightarrow (+1, +1) \quad 1 \rightarrow (+1, -1) \quad 2 \rightarrow (-1, -1) \quad 3 \rightarrow (-1, +1)$$

That is, each element in $\mathbb{Z}_4$ is replaced with a pair of -1 or +1. The resulting two-level design, denoted by $D = \phi(C)$, is called the binary image of $C$. Note that $C$ is a $4^N \times l$ matrix over $\mathbb{Z}_4$ and $D$ is a $4^N \times 2l$ matrix over $\mathbb{Z}_2$.

The following example illustrates the construction1 of a two-level design from a
generator matrix over \( Z_4 \). We consider a \( 2 \times 4 \) generator matrix \( G \)

\[
G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}
\]

All linear combinations of the two rows of \( G \) form a \( 16 \times 4 \) linear code \( C \) over \( Z_4 \). Applying the Gray map, a \( 16 \times 8 \) binary image \( D = \phi(C) \) is obtained. The results are shown below.

\[
C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 0 & 3 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 3 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 0 & 3 & 3 \\ 3 & 1 & 0 & 1 \\ 3 & 2 & 1 & 3 \\ 3 & 3 & 2 & 1 \end{bmatrix},
\]

\[
D = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\ +1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 \\ -1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 \\ -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}
\]

The simplest way to indicate the generator matrix is by the notation of column indexes. Each column in \( G \) is represented by a number equal to \( \sum_{i=1}^{N_i} g_i 4^{i-1} \). For example, the column indexes of the above \( G \) are \((4,1,5,6)\).

### 3. Main Results

We consider the generator matrix \( G \) with the combined structure of an \( N \times N \) identity matrix \( I_N \) and an additional \( N \times 1 \) column \( v \) as follow.

\[
G = [I_N \ v]
\]

Since the identity matrix generates a complete design, the property of the design depends on the columns \( v \), which is characterized by a count vector,

\[
[c_0 \ c_1 \ c_2 \ c_3],
\]

where \( c_i \) is the number of rows in \( v \) that the entry is “i”.

Theorem 1 characterizes the type of the designs and specifies the number of words and their wordlengths.

**THEOREM 1.** Given a generator matrix \( G = [I_N \ v] \).

(i) If \( c_1 + c_3 = 0 \), then the design is regular and has 1 complete word of length 2 and 2 complete words of length \( 2c_2 + 1 \);

(ii) If \( c_1 + c_3 = 1 \), then the design is regular and has 1 complete word of length 4 and 2 complete words of length \( 2c_2 + 2 \);
If $c_1 + c_3 > 1$, then the design is non-regular and has 1 complete word of length $k_2 = 2c_1 + 2c_3 + 2$ and $2/\rho^2$ partial words of length $k_1 = c_1 + 2c_2 + c_3 + 1$ with aliasing index $\rho = 2^{-\frac{c_1+c_3}{2}}$.

Instead of going through tedious computations of the linear combination and the Gray map transformation, Theorem 1 suggests a shortcut of characterization and counting directly from the generator matrix directly. This leads to the main results on design optimality in terms of resolution and aberration stated in the next theorem.

The following theorem gives formulas to calculate the generalized resolution, generalized wordlength pattern and projectivity of a design. The only information required in those formulas is the count vector $C$ of the additional $Z_4$ column:

**THEOREM 2.** Given a generator matrix $G$ of a design $D$ as $G = [I_N \ v]$. Let $k_1 = c_1 + 2c_2 + c_3 + 1$, $k_2 = 2c_1 + 2c_3 + 2$ and $\rho = 2^{-\frac{c_1+c_3}{2}}$.

(i) The generalized resolution of $D$ is
   \[
   GR(D) = \min(k_1, k_2) + (1 - \rho)I_{k_1 < k_2}
   \]

(ii) The elements in the generalized wordlength pattern of $D$ is
   \[
   A_k(D) = \begin{cases} 
0 & k \neq k_1 \neq k_2 \\
1 & k = k_2 \neq k_1 \\
2 & k = k_1 \neq k_2 \\
3 & k = k_1 = k_2 
\end{cases}
   \]

(iii) The projectivity of $D$ is
   \[
   p(D) = \begin{cases} 
2k - 1 & c_1 + c_3 < N \\
2k - 3 & c_1 + c_3 = N 
\end{cases}
   \]

An application of Theorem 2 leads to is the optimal design construction method under maximum generalized resolution criterion, generalized minimum aberration (GMA) criterion and maximum projectivity criterion. It is stated in the next theorem.

**THEOREM 3.** Given a generator matrix $G = [I_N \ v]$.

(i) If $N = 3k - 1$, $k \geq 1$, then the optimal design under Max. GR and GMA criteria must have $c_0 = 0$, $c_1 + c_3 = \frac{2N-1}{3}$ and $c_2 = \frac{N+1}{3}$. This design has $GR = \frac{4}{3}N + \frac{4}{3}$ and $A_{GR} = 3$;

(ii) If $N = 3k \forall k \in \mathbb{Z}$, then the optimal design under Max. GR and GMA criteria must have $c_0 = 0$, $c_1 + c_3 = \frac{2N}{3}$ and $c_2 = \frac{N}{3}$. This design has $GR = \frac{4}{3}N + 2 - 2^{\frac{N}{3}}$.
\( A_{GR} = 2 \) and \( A_{GR+1} = 1 \);

(iii) If \( N = 3k + 1 \forall k \in \mathbb{Z} \), then the optimal design under Max. GR criterion must have \( c_0 = 0, \ c_1 + c_3 = \frac{2N+1}{3} \) and \( c_2 = \frac{N-1}{3} \). This design has \( GR = \frac{4}{3}N + \frac{5}{3} - 2 \frac{N-1}{3} \), \( A_{GR} = 2 \) and \( A_{GR+1} = 1 \);

(iv) If \( N = 3k + 1 \forall k \in \mathbb{Z} \), then the optimal design under GMA criterion must have \( c_0 = 0, \ c_1 + c_3 = \frac{2N-2}{3} \) and \( c_2 = \frac{N+2}{3} \). This design has \( GR = \frac{4}{3}n + \frac{2}{3} \ A_{GR} = 1 \) and \( A_{GR+1} = 2 \);

(v) The optimal design to maximize its projectivity must have \( c_1 + c_3 = N - 1 \) or \( N \). This design has \( p = 2N - 1 \).

A new class of optimal designs, under any of the three criteria mentioned above, is constructed from quaternary codes suggested in Theorem 3. A list of optimal designs, including their column selections, generalized resolution, generalized wordlength pattern and projectivity, is provided at the end of this section.

Theorem 4 suggests a stronger result that some of the quaternary-code designs are optimal over all possible designs.

**THEOREM 4.** Among all designs with \( 2N + 2 \) columns and \( 4^N \) runs.

(i) The designs with generalized minimum aberration among all quaternary-code designs have generalized minimum aberration among all possible designs.

(ii) The designs with maximum projectivity among all quaternion-code designs have maximum projectivity among all possible designs.

The following table provides the optimal designs calculated from Theorem 3. They are arranged in the ascending order of their number of runs listed in the first column. In the second column, the first number is the number of columns in the design, and “-2” is a notation of the fractional factorial designs with only a quarter of the total runs in the corresponding full factorial designs. The alphabets at the end denote which criteria the design fulfills. In particular, “r” represents the maximum generalized resolution criterion, “a” represents the generalized minimum aberration criterion and “p” represents the maximum projectivity criterion.

The third column is the column index of the additional column \( v \) in the generator matrix \( G \). The next three columns, under the category “quaternary-code designs”, are the generalized wordlength pattern (GWLP), generalized resolution (GR) and projectivity (p) of the designs constructed from the generator matrix \( G = [I_N \ v] \). The last two columns, under the category “Best Regular Design”, are the resolution (R) and the projectivity (p) of the optimal
### TABLE OF OPTIMAL DESIGNS:

<table>
<thead>
<tr>
<th>(4^n) Runs</th>
<th>Design</th>
<th>(v)</th>
<th>Quaternary-code Designs</th>
<th>Best Regular Designs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>GWLP</td>
<td>GR</td>
</tr>
<tr>
<td>(4^3)</td>
<td>8-2rap</td>
<td>22</td>
<td>002100</td>
<td>5.5</td>
</tr>
<tr>
<td>(4^4)</td>
<td>10-2p</td>
<td>86</td>
<td>00020000</td>
<td>6.5</td>
</tr>
<tr>
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<td>90</td>
<td>00012000</td>
<td>6.0</td>
</tr>
<tr>
<td>(4^5)</td>
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<td>0000030000</td>
<td>8.0</td>
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</tr>
<tr>
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<td>12.9375</td>
</tr>
</tbody>
</table>

regular designs with the same runs.

Readers who are interested in the proofs of the above theorems are referred to the next section for details.

### 4. Proofs

Some additional definitions and technical lemmas are introduced prior to the proofs of the theorems.

#### 4.1 Definitions and Notations

We consider a design \(D\) of \(n\) runs and \(m\) factors. Instead of summing up the product of each row to obtain \(J\)-characteristics, we obtain a vector with length equal to the run size. We call this vector to be the interaction product of \(s\), is

\[
IP\{s\} = c_1 * c_2 * \cdots * c_k = \begin{pmatrix}
  c_{11} \cdots c_{1k} \\
  c_{21} \cdots c_{2k} \\
  \cdots \\
  c_{N1} \cdots c_{Nk}
\end{pmatrix}
\]

where \(c_{ij}\) is the \(i\)th entry of column \(c_j\), and \(*\) is the Hadamard product operator. The sum of
all entries in the interaction product is equivalent to the $J$-characteristics. The interaction product consists of two structures: a balanced structure and an unit structure. A balanced structure is defined to be the entries containing equal numbers of -1 and +1, and a unit structure is defined to be the remaining entries of the interaction product. If the unit structure has all “+1” entries, it is a positive unit structure. If it has all “-1” entries, it is a negative unit structure.

Consider an $N \times (N + 1)$ generator matrix:

$$G = [I_N \ v]$$

where $I_N$ is an $N \times N$ identity matrix and $v$ is a $Z_4$ column. A non-split of a $Z_4$ column $v$ in $G$, denoted by $v_{NS}$, is a matrix of two binary columns in $D$ generated from $v$. A split of $v$ in $G$, denoted by $v_S$, is the one of these two binary columns $v_{NS}$. A split-left of $v$, denoted by $v_{S-}$, is the first binary column of $v_{NS}$ while a split-right of $v$, denoted by $v_{S+}$, is the second binary column of $v_{NS}$. We say that split-left is the inverse of split-right, and vice versa, and we denote the inverse of $v_S$ to be $v_S$.

Here are some simple properties of designs constructed from quaternary codes.

**PROPERTY 1.** $IP\{v_S, v_S\} = 1^T$ where 1 represents a row whose entries are all “+1”.

**PROPERTY 2.** $IP\{v_S, v_{NS}\} = v_S$

**PROPERTY 3.** $(U \ W)^T_S = (U \ W + 2)^T_S$ where $u_i \in \{0,2\}$ and $w_i \in \{1,3\}$

**PROPERTY 4.** $(0 \ Y)^T_S = [(Y)_S \ (Y)_S \ (Y)_S \ (Y)_S]^T$ where $y_i \in Z_4$

The first property states that the interaction product of a $Z_4$ column and itself returns a column with a unit structure. The second property states that the interaction product of a split of a $Z_4$ column and a non-split of the same $Z_4$ column returns the inverse of split of the $Z_4$ column. The third property states that (a) The split of a $Z_4$ column with some even entries, $u_i$, is equivalent to the inverse of split of the $Z_4$ column with the same even entries. (b) The split of a $Z_4$ column with some odd entries, $w_i$, is equivalent to the inverse of split of the $Z_4$ column with the odd entries increased by 2 (modulo 4). It can be verified from the Gray map transformation, where $0_S = 0_{\bar{S}}$, $1_S = 3_{\bar{S}}$, $2_S = 2_{\bar{S}}$, $3_S = 1_{\bar{S}}$. The forth property states that the split of a $Z_4$ column with “0” as the first entry is equivalent to a quadruple of the split of the same $Z_4$ column without “0”.

**4.2 Technical Lemmas**

There are some technical lemmas on the results of the interaction products of the subset, which include either the split or the non-split of $v$ and some identity columns. They are the building blocks of the Theorem 1.
Lemma 1 and Lemma 2 provide the results associated to $v = (2 \ Y)^T$ while Lemma 3 to Lemma 5 provide the results associated to $v = (1 \ Y)^T$, where $y_i \in Z_4$. The results associated to $v = (3 \ Y)^T$ is analogue to those associated to $v = (1 \ Y)^T$ with the inverse of the splits, and the results associated to $v = (0 \ Y)^T$ can be reduced to any one of the three cases above after applying Property 4.

**LEMMA 1.** $IP\{(2 \ Y)^T_S, (1 \ 0)^T_{NS}, (0 \ Y)^T_S\} = 1^T$ where $y_i \in Z_4$

**PROOF.** We consider the construction procedures of the quaternary code designs, and consider the above 4 columns are a subset of a generating matrix. Following the step of forming the 4-level design C, we have

$$G = (2 \ Y)^T \to C = 2 \times \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T + \begin{bmatrix} Y_C & Y_C & Y_C & Y_C \end{bmatrix}^T$$

$$G = (0 \ Y)^T \to C = 0 \times \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T + \begin{bmatrix} Y_C & Y_C & Y_C & Y_C \end{bmatrix}^T$$

where $Y_C^T$ is a 4-level row derived from generator matrix $G = (Y)^T$. In modulo 4, $2 \times \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix}$.

Based on the fact that the binary image of $\begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix}$ has the opposite sign of $\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$, then

$$(2 \ Y)^T_S = [Y_{S-} + 0 \ Y_{S-} + 2 \ Y_{S-} + 0 \ Y_{S-} + 2]^T = [Y_{S-} - Y_{S-} \ Y_{S-} - Y_{S-}]^T$$

$$(0 \ Y)^T_S = [Y_{S-} + 0 \ Y_{S-} + 0 \ Y_{S-} + 0 \ Y_{S-} + 0]^T = [Y_{S-} \ Y_{S-} \ Y_{S-} \ Y_{S-}]^T$$

Therefore, the interaction product of $(2 \ Y)^T_S$ and $(0 \ Y)^T_S$ is

$$IP\{(2 \ Y)^T_S, (0 \ Y)^T_S\} = [1 \ -1 \ 1 \ -1]^T$$

On the other hands, the interaction product of $(1 \ 0)^T_{NS}$ is

$$IP\{(1 \ 0)^T_{NS}\} = [1 \ 1 \ -1 \ -1]^T \ast [1 \ -1 \ -1 \ 1]^T = [1 \ -1 \ 1 \ -1]^T$$

Since both interaction products are equal, Property 1 suggests that the interaction product of $(2 \ Y)^T_S$ and $(1 \ 0)^T_{NS}$ is a column with unit structure. The case of for the split-right of the column is analogue to the above proof.

For example, the interaction product of a subset $D_{sub}$ consisting of $(2 \ 1 \ 1)^T_S$, $(1 \ 0 \ 0)^T_{NS}$ and $(0 \ 1 \ 1)^T_S$ is equivalent to the column with a positive unit structure.

**LEMMA 2.** $IP\{(2 \ Y)^T_{NS}, (0 \ Y)^T_{NS}\} = 1^T$ where $y_i \in Z_4$

**PROOF.** In Lemma 1, the interaction product of the split of these two columns is

$$IP\{(2 \ Y)^T_S, (0 \ Y)^T_S\} = IP\{(2 \ Y)^T_{S+}, (0 \ Y)^T_{S+}\} = [1 \ -1 \ 1 \ -1]$$

Then, if we consider the interaction product of the non-split of these two columns,

$$IP\{(2 \ Y)^T_{NS}, (0 \ Y)^T_{NS}\} = [1 \ -1 \ 1 \ -1]^T \ast [1 \ -1 \ 1 \ -1]^T = 1^T$$

In other words, $IP\{(2 \ Y)^T_{NS}\} = IP\{(0 \ Y)^T_{NS}\}$. □

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For example, the interaction product of a subset $D_{sub}$ consisting of $(2 \ 1 \ 1)^T_{NS}$ and $(0 \ 1 \ 1)^T_{NS}$ is equivalent to the column with a positive unit structure.

**Lemma 3.** $IP((1 \ Y)^T_{NS}, (1 \ 0)^T_{NS}, (0 \ Y)^T_{NS}) = \pm 1^T$ where $y_i \in Z_4$.

**Proof.** We consider a general $1 \times (n+1)$ $Z_4$ column containing a “1” in the $1^{st}$ entry, i.e. $(1 \ Y)^T$ where $Y^T$ is a column vector of length $n$ with entries $u_i \in Z_4$.

$$IP((1 \ Y)^T_{NS}, (1 \ 0)^T_{NS}, (0 \ Y)^T_{NS}) = \pm 1^T$$

To show this, we consider the construction procedures of the quaternary code designs, and consider the above 4 columns are a subset of a generating matrix. Following the step of forming the 4-level design $C$, we have

$$G = (1 \ Y)^T \rightarrow C = 1 \times [0 \ 1 \ 2 \ 3]^T + [Y_C \ Y_C \ Y_C \ Y_C]^T$$

$$G = (0 \ Y)^T \rightarrow C = 0 \times [0 \ 1 \ 2 \ 3]^T + [Y_C \ Y_C \ Y_C \ Y_C]^T$$

where $Y_C^T$ is a 4-level row derived from generator matrix $G = (Y)^T$. Based on the construction of quaternary code designs, the 4-level $Y^T$ is a column built by 4 possible orientations:$[0 \ 1 \ 2 \ 3]^T$, $[1 \ 2 \ 3 \ 0]^T$, $[2 \ 3 \ 0 \ 1]^T$ and $[3 \ 0 \ 1 \ 2]^T$. Then the interaction product of $(1 \ U)^T_{NS}$, in the form of binary image, is either $[+1 \ -1 \ +1 \ -1]^T$ or $[-1 \ +1 \ -1 \ +1]^T$. In addition, for any $Y_i$, $IP((Y_i + 1NS)^T = \pm 1^T)$. Therefore, we may obtain

$$IP((1 \ Y)^T_{NS}) = \pm [IP(Y^T_{NS}) \ -IP(Y^T_{NS}) \ IP(Y^T_{NS}) \ -IP(Y^T_{NS})]^T$$

$$IP((0 \ Y)^T_{NS}) = [IP(Y^T_{NS}) \ IP(Y^T_{NS}) \ IP(Y^T_{NS}) \ IP(Y^T_{NS})]^T$$

The interaction product of these columns is

$$IP((1 \ Y)^T_{NS}, (0 \ Y)^T_{NS}) = \pm [1 \ -1 \ 1 \ -1]^T$$

From Lemma 1, we have shown that

$$IP((1 \ 0)^T_{NS}) = [1 \ -1 \ 1 \ -1]^T$$

This implies that the interaction product of

$$IP((1 \ Y)^T_{NS}, (1 \ 0)^T_{NS}, (0 \ Y)^T_{NS}) = \pm 1^T$$

For example, the interaction product of a subset $D_{sub}$ consisting of $(1 \ 3 \ 2)^T_{NS}$, $(1 \ 0 \ 0)^T_{NS}$ and $(0 \ 3 \ 2)^T_{NS}$ is equivalent to the column with a unit structure.

**Lemma 4.** $IP((1_n)^T_{NS}) = B^T_{n}$ where $1_n$ is a row vector of “1” with length $n$ and $B_n$ is a row vector called a B-sequence of order $n$, which is defined as:

$$B_0 = 1, \ B_{t+1} = [+B_i \ -B_i \ +B_i \ -B_i]$$

**Proof.** By induction. It is simple to show that $IP((1_1)^T_{NS}) = B^T_1 = [+ \ - \ + \ -]^T$. Assume $n=k$ is true. Then for $n=k+1$, we have $IP((1_{k+1})^T_{NS}) = IP((1 \ 1_k)^T_{NS})$. Lemma 3 suggests that it can be express as $IP((1_{k+1})^T_{NS}) = IP((1 \ 0)^T_{NS}) \ast IP((0 \ 1_k)^T_{NS})$. Using...
the assumption of $n=k$ and Property 4,

$$IP\{1_{k+1}\}_S^T = [+1 \ -1 \ +1 \ -1] \ast [B_k \ B_k \ B_k] = [+B_k \ -B_k \ +B_k \ -B_k]$$

The final expression is defined as $B_{k+1}$.

For example, the interaction product of a subset $D_{sub}$ consisting of $(1 \ 1)_{\text{NS}}^T$ is equivalent to $B_2 = [+B_1 \ -B_1 \ +B_1 \ -B_1]$

$$= [+1 \ -1 \ +1 \ -1 \ +1 \ -1 \ +1 \ -1 \ +1 \ -1 \ +1]$$

**Lemma 5.** $IP\{(1 \ 1_n)_S^T, (1 \ 0)_S^T, (0 \ 1_n)_S^T\} = [1_{4n} \ B_n \ 1_{4n} \ B_n]^T$ where $1_n$ is a row vector of $“1”$ with length $n$ and $B_n$ is a $B$-sequence of order $n$.

**Proof.** By induction. For $m=2$, $B_1 = [+ - + -]$. Then it is simple to show that $IP\{(1 \ 1)_{\text{NS}}^T\} \ast [1 \ B_1 \ 1 \ B_1]^T$ is equivalent to $IP\{(1 \ 0)_{\text{NS}}^T, (0 \ 1)_{\text{NS}}^T\}$, which is $[+ + - - + + - - - - + + - - + +]^T$. So it is true for the case of $m=2$.

Assume the case of $m=l$ is true. For $m=l+1$, we start from $IP\{(1_{l+1})_{\text{NS}}^T\} \ast [1_l \ B_l \ 1_l \ B_l]^T$. If we divide $1_{l+1}$ into 4 segments with equal lengths and substitute each of them with the case of $m=l$, the result of the multiplication returns

$$\begin{cases}
IP\{(1 \ 0_{l-1})_{\text{NS}}^T, (0 \ 1_{l-1})_{\text{NS}}^T\} \ast [1_{l-1} \ B_{l-1} \ 1_{l-1} \ B_{l-1}]^T \
IP\{(1 \ 0_{l-1})_{\text{NS}}^T, (0 \ 1_{l-1})_{\text{NS}}^T\} \ast [1_{l-1} \ B_{l-1} \ 1_{l-1} \ B_{l-1}]^T 
\end{cases} \ast \begin{bmatrix} 1_l^T \\ B_l^T \\ 1_l^T \\ B_l^T \end{bmatrix}$$

If we extract the signs as a vector and apply a variation of Lemma 4 that $IP\{(1 \ 0_{n})_{\text{S}+}^T, (0 \ 1_n)_{\text{S}+}^T\} = IP\{(1 \ 0_{n})_{\text{S}+}^T, (0 \ 1_n)_{\text{S}+}^T\} \ast B_{n+1}^T$, we may rewrite the above expression in the form of

$$\begin{bmatrix} +1_l^T \\ +1_l^T \\ -1_l^T \\ -1_l^T \end{bmatrix} \ast \begin{bmatrix} IP\{(1 \ 0_{l-1})_{\text{S}+}^T, (0 \ 1_{l-1})_{\text{S}+}^T\} \ast [1_{l-1} \ B_{l-1} \ 1_{l-1} \ B_{l-1}]^T \\ IP\{(1 \ 0_{l-1})_{\text{S}+}^T, (0 \ 1_{l-1})_{\text{S}+}^T\} \ast [1_{l-1} \ B_{l-1} \ 1_{l-1} \ B_{l-1}]^T \\
-IP\{(1 \ 0_{l-1})_{\text{S}+}^T, (0 \ 1_{l-1})_{\text{S}+}^T\} \ast [1_{l-1} \ B_{l-1} \ 1_{l-1} \ B_{l-1}]^T \\
-IP\{(1 \ 0_{l-1})_{\text{S}+}^T, (0 \ 1_{l-1})_{\text{S}+}^T\} \ast [1_{l-1} \ B_{l-1} \ 1_{l-1} \ B_{l-1}]^T 
\end{bmatrix} \ast \begin{bmatrix} 1_l^T \\ B_l^T \\ 1_l^T \\ B_l^T \end{bmatrix}$$

It can further simplify into $IP\{(1 \ 0_l)_{\text{S}+}^T, (0 \ 1_l)_{\text{S}+}^T\}$ due to the assumption of the case of $m=l$. The proof for the split-right is the analogue of the proof for the split-left shown above.

For example, the interaction product of a subset $D_{sub}$ consisting of $(1 \ 1 \ 1)_{S}^T$, $(1 \ 0 \ 0)_{S}^T$ and $(0 \ 1 \ 1)_{S}^T$ is equivalent to the column with the structure of $[1 \ B_2 \ 1 \ B_2]^T$, where $1$ is a row of $1$ and $B_2$ is given in previous example.

Lemma 6 is a tool to calculate the aliasing index of a design constructed from a special kind of generator matrix $G$, with all “1” in the last column. It bypasses the tedious calculation of $J$-characteristics and it provides the aliasing index solely depends on the number of “1” in the last column.
LEMMA 6. Given \( G = [I_N \ v] \) where \( v = 1^T_N \). Consider two subsets \( s_1 = \{v_{s-1}, x_3, \ldots, x_N\} \) and \( s_2 = \{v_{s+}, x_3, \ldots, x_N\} \), where \( x_k \) is either split-left or split-right of the \( k \)th identity column. Then there are \( 2/\rho^2 \) partial words such that (a) for \( N = 2t - 1 \), either \( \rho(s_1) = 2^{N-1} \) and \( \rho(s_2) = 0 \), or \( \rho(s_1) = 0 \) and \( \rho(s_2) = 2^{N-1} \); (b) for \( N = 2t \), \( \rho(s_1) = \rho(s_2) = 2^{-N/2} \).

PROOF. By induction. It is trivial for \( N=2 \) and \( N=3 \) to be true. Assume \( N=k \) is true. Then for \( N=k+1 \), we label the columns of our binary image of the design \( D \) as follows:

\[
D = [v_{k+1} l_{k+1}] = [1 \ 2 \ 3 \ 4 \ 5 \ \ldots \ 2k + 4]
\]

Since the last \( 2k+2 \) columns is complete, we may consider only the \( J \)-characteristics associated with the last two columns, which are

\[
\begin{align*}
J_{k+1}(1, 3, 5 \ldots, 2k + 4) &= 2J_k(3, 5 \ldots, 2k + 4) + 2J_k(4, 5 \ldots, 2k + 4) \\
J_{k+1}(1, 4, 5 \ldots, 2k + 4) &= 2J_k(3, 5 \ldots, 2k + 4) - 2J_k(4, 5 \ldots, 2k + 4) \\
J_{k+1}(2, 3, 5 \ldots, 2k + 4) &= -2J_k(3, 5 \ldots, 2k + 4) + 2J_k(4, 5 \ldots, 2k + 4) \\
J_{k+1}(2, 4, 5 \ldots, 2k + 4) &= 2J_k(3, 5 \ldots, 2k + 4) + 2J_k(4, 5 \ldots, 2k + 4)
\end{align*}
\]

If \( k \) is odd, \( J_{k+1} = \pm 2J_k \), then \( \rho_{k+1} = \frac{\pm 2J_k}{4N} = \pm \frac{1}{2} \rho_k \), so \( \rho(s_1) = \rho(s_2) = 2^{-(k+1)/2} \). There are \( 2^{k+1} \) different combinations in both \( s_1 \) and \( s_2 \). Since the combinations in both subsets form the partial words, the number of partial words is \( 2^{k+2} \).

If \( k \) is even, \( J_{k+1} = \pm 4J_k \) or \( J_{k+1} = 0 \), then \( \rho_{k+1} = \frac{\pm 4J_k}{4N} = \pm \rho_k \), so either \( \rho(s_1) = 2^{-k/2} \) and \( \rho(s_2) = 0 \), or \( \rho(s_1) = 0 \) and \( \rho(s_2) = 2^{-k/2} \). There are \( 2^{k+1} \) different combinations in both \( s_1 \) and \( s_2 \). Since the combinations in either one of the subsets form the partial words, the number of partial words is \( 2^{k+1} \).

Lemma 7 is a similar tool to calculate the projectivity of a design constructed from a generator matrix \( G \) with all “1” in the last column. It is similar to Lemma 6 because it also bypasses the tedious calculation of \( J \)-characteristics and provides the projectivity solely depends on the number of “1” in the last column.

LEMMA 7. Given \( G = [I_N \ v] \) where \( v = 1^T_N \). Then the projectivity of \( D \) constructed from \( G \) is \( p = 2N + 1 \).

PROOF. By induction. It is trivial for \( N=1 \) and \( N=2 \) to be true. Assume \( N=k \) is true. Then for \( N=k+1 \), we may express our binary image of the design \( D \) in the same form as in the proof of Lemma 6. Instead of the first \( 2k+2 \) columns, we are able to view the last \( 2k+2 \) columns as a full \( 2^{2k+2} \) design. Then we consider three different cases of including any of the first two
columns: (i) None of \( \{3, 4\} \in s \). It is trivial that \( s \) is a full \( 2^{2k+2} \) design because \( s \) containing \( \{1, 2, 5, \ldots, 2k+4\} \) is isomorphic to \( s \) containing \( \{3, 4, 5, \ldots, 2k+4\} \); (ii) Both \( \{3, 4\} \in s \). The projection onto \( s \) contains a full \( 2^{k+1} \) design; (iii) Any one of \( \{3, 4\} \in s \). If there are more than three columns, the subset still contains at least one non-split of the column \( \{2i-1, 2i\} \) after deleting any two columns plus any one of \( \{3, 4\} \). Since \( s \) containing \( \{2i-1, 2i, 3, \text{ others}\} \) is isomorphic to \( s \) containing \( \{2i-1, 3, 4, \text{ others}\} \), and similar isomorphism is between \( \{2i-1, 2i, 4, \text{ others}\} \) and \( \{2i, 3, 4, \text{ others}\} \). (iii) reduced to (ii).

\[ \square \]

### 4.3 Proofs of Theorems

**Proof of Theorem 1.** Assume we have the generator matrix with structure \( G = [I_N \ v] \) where the count matrix of \( v \) is \( C = [c_0 \ c_1 \ c_2 \ c_3] \). Since \( I_N \) is a complete matrix, the formation of words, both partial and complete words, is associated to either the splits or non-split of \( v \). Notice that the wordlength is equivalent to the number of columns in the subset such that the interaction product of the subset returns an unbalanced structure.

**Wordlengths for Non-Splits:**

First, we rearrange the rows such that the row with “1” and “3” in \( v \) are the first \( c_1+c_2 \) rows, and the rows with “2” in \( v \) are the next \( c_2 \) rows. By Lemma 2, \( IP\{(1 \ Y)_{NS}^T, (0 \ Y)_{NS}^T\} = \pm 1^T \), where \( y_i \in Z_4 \). By repeating the reduction of “1” and “3” \( c_1 + c_3 \) times, we are able to reduce the expression to \( (0 \ U)_{NS}^T \), where \( u_i \in \{0,2\} \). Then by Property 3, \( (2 \ 0)^{T}_{S-} = (2 \ 0)^{T}_{S+} \). Also by Property 1, \( IP\{(2 \ 0)^{T}_{S-}, (2 \ 0)^{T}_{S+}\} = IP\{(2 \ 0)^{T}_{S-}, (2 \ 0)^{T}_{S-}\} = 1^T \). This leads to the following 3 results: When \( c_1 + c_3 = 0 \), it returns a complete word of length 2; When \( c_1 + c_3 = 1 \), it returns a complete word of length 4; When \( c_1 + c_3 > 1 \), it returns a complete word of length \( 2c_1 + 2c_3 + 2 \).

**Wordlength for Splits:**

First, note that the split of “3” is equivalent to the inverse of split of “1” by Property 3. Then we rearrange the rows such that the rows with “2” in \( v \) are the first \( c_2 \) rows and the rows with “1” in \( v \) are the last row. By Lemma 1, \( IP\{(2 \ U \ 1)_{S-}, (0 \ 0)_{NS}\} = (0 \ U \ 1)_{S-}\). We repeat \( c_2 \) times on the process of this transformation, and it returns \( (0 \ 0 \ 1)^{T}_{S-} \). When \( c_1 + c_3 = 0 \), it is a column of unit structure, so it results a complete word of length \( 1 + 2c_2 \). When \( c_1 + c_3 = 1 \), it is \( (0 \ 1)^{T}_{S-} \), so it results a complete word of length \( 1 + 2c_2 \). When \( c_1 + c_3 > 1 \), it is \( (0 \ 1)^{T}_{S-} \), which requires a further treatment by Lemma 4 such that \( IP\{(1 \ 1_{N-1})_{S-}^T, (1 \ 0)_{S-}^T, (0 \ 1_{N-1})_{S-}^T\} = [1 \ B_{N-1} \ 1 \ B_{N-1}]^T \). We repeat \( c_1 + c_3 \) times
on the process of this transformation, and it returns an unbalanced structure with some $1^T$ and B-sequence. This results a partial word of length $c_1 + 2c_2 + c_3 + 1$.

**Aliasing Index and Number of Partial Words in case (iii):**

Since the existence of “2” in $v$ can be viewed as “0” through the treatment of Lemma 1, and according to Property 4, the existence of a “0” repeats the rest of the structure by four times, the aliasing index is only related to the number of “1” and “3” in $v$. In addition, the existence of “3” is equivalent to the inverse of split of “1”. Therefore, after some row rearrangements and switching the split to eliminate the existence of “3”, we are able to have $v$ such that the last $c_1+c_3$ rows are all “1”. Then the aliasing index and the number of partial words are calculated from Lemma 6.

**Proof of Theorem 2.** (i) It is the same as the definition of the generalized resolution. In particular, the whole-number term is to select the word with the shorter length, and the decimal term is the definition with the substitution of the aliasing index. If the partial word is shorter, the decimal term is added to the generalized resolution. Otherwise, the decimal term disappears because the aliasing index of a complete word is 1. (ii) If there exists a complete word of length $k = k_2$ then according to the definition of the wordlength pattern, $A_k = 1$. If there exists $2/p^2$ partial words of length $k = k_1$, then $A_k = (p^2)(2/p^2) = 2$. If the complete word and the partial words have the same lengths, then $A_k = 1 + 2 = 3$. (iii) The existence of “2” and “0” does not affect the projectivity because the existence of “2” in $v$ can be viewed as “0” through the treatment of Lemma 1, and according to Property 4, the existence of a “0” repeats the rest of the structure by four times, which by definition of projectivity, is viewed as the repetition. In addition, the existence of “3” is equivalent to the inverse of split of “1”. Therefore, after some row rearrangements and switching the split to eliminate the existence of “3”, we are able to have $v$ such that the last $c_1+c_3$ rows are all “1”. Then the projectivity of a design follows from Lemma 7.

**Proof of Theorem 3.** We assume $c_0 = c_3 = 0$, so we only need to consider $c_1$ and $c_2 = N - c_1$. Then the generalized resolution and the wordlength pattern can be calculated by Theorem 1 and Theorem 2.

**Maximum GR and GMA when $N = 3x - 1$:**

Consider $k_1 \geq k_2$, then $c_1 \leq \frac{2N-1}{3}$ and $c_2 \geq \frac{N+1}{3}$. This returns $k_1 = \frac{4N+4}{3}$ and $k_2 = \frac{4N+4}{3}$ when $c_1 = \frac{2N-1}{3}$ and $c_2 = \frac{N+1}{3}$. It implies that $GR = \frac{4}{3}N + \frac{4}{3}$ and $A_{GR} = 3$. Consider
$k_1 < k_2$, then $c_1 > \frac{2N-1}{3}$ and $c_2 < \frac{N+1}{3}$. This returns $k_1 = \frac{4N+1}{3}$ and $k_2 = \frac{4N+10}{3}$ when $c_1 = \frac{2N+2}{3}$ and $c_2 = \frac{N-2}{3}$. It implies that $GR = \frac{4}{3}N + \frac{4}{3} - \rho$ and $(A_{GR} = 2, A_{GR+1} = 1)$. Therefore, the case $k_1 \geq k_2$ has maximum GR and GMA.

**Maximum GR and GMA when $N = 3x$:**

Consider $k_1 \geq k_2$, then $c_1 \leq \frac{2N-1}{3}$ and $c_2 \geq \frac{N+1}{3}$. This returns $k_1 = \frac{4N+4}{3}$ and $k_2 = \frac{4N}{3}$ when $c_1 = \frac{2N-3}{3}$ and $c_2 = \frac{N+3}{3}$. It implies that $GR = \frac{4}{3}N$ and $(A_{GR} = 1, A_{GR+1} = 2)$.

Consider $k_1 < k_2$, then $c_1 > \frac{2N-1}{3}$ and $c_2 < \frac{N+1}{3}$. This returns $k_1 = \frac{4N+3}{3}$ and $k_2 = \frac{4N+6}{3}$ when $c_1 = \frac{2N}{3}$ and $c_2 = \frac{N}{3}$. It implies that $GR = \frac{4}{3}N + 2 - \rho$ and $(A_{GR} = 2, A_{GR+1} = 1)$. Therefore, the case $k_1 < k_2$ has maximum GR and GMA.

**$c_1$ when $N = 3x + 1$:**

Consider $k_1 \geq k_2$, then $c_1 \leq \frac{2N}{3}$ and $c_2 \geq \frac{N}{3}$. This returns $k_1 = \frac{4N+5}{3}$ and $k_2 = \frac{4N+2}{3}$ when $c_1 = \frac{2N-2}{3}$ and $c_2 = \frac{N+2}{3}$. It implies that $GR = \frac{4}{3}N + \frac{2}{3}$ and $(A_{GR} = 1, A_{GR+1} = 2)$.

Consider $k_1 < k_2$, then $c_1 > \frac{2N}{3}$ and $c_2 < \frac{N}{3}$. This returns $k_1 = \frac{4N+2}{3}$ and $k_2 = \frac{4N+8}{3}$ when $c_1 = \frac{2N+1}{3}$ and $c_2 = \frac{N-1}{3}$. It implies that $GR = \frac{4}{3}N + \frac{5}{3} - \rho$ and $(A_{GR} = 2, A_{GR+1} = 1)$. Therefore, the case $k_1 < k_2$ has maximum GR and the case $k_1 \geq k_2$ has GMA.

**Maximum Projectivity Criterion:**

Since the calculation of projectivity is independent of the value of $c_0$ and $c_2$, maximizing $c_1 + c_3$ is equivalent to maximizing projectivity. Therefore, it is reasonable to set $c_1 + c_3 = N$ and $c_0 = c_2 = 0$. However, if we set $c_0 = 0$, $c_1 + c_3 = N - 1$ and $c_2 = 1$, the formula returns same projectivity. The later one is preferred because under the same level of projectivity, it has comparably higher resolution than the prior one.

$\square$
PROOF OF THEOREM 4. (i) It follows Theorem 2 in Xu (2005). (ii) By contradiction. For our binary image $D$ with dimension $2^{2N} \times 2(N + 1)$, assume $p = 2N$ exists, then $D = OA(2^{2N}, 2N + 2, 2N)$. However, it does not exist according to the first inequality of Theorem 2.19 in Hedayat et. al. (1999), where $s=2$, $t=2N$ and $k=2N+2$, so $p = 2N$ does not exist.

\[\square\]

REFERENCE


