CONSTRUCTION OF OPTIMAL MULTI-LEVEL SUPERSATURATED DESIGNS

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Outline

- Introduction
- Optimality criteria
- Optimal results
- Construction
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- Mixed-level designs
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Introduction

A supersaturated design (SSD)

- run size is not large enough for estimating all the main effects
- can study more factors than number of runs
- assuming effect sparsity principle [Box and Meyer (1986)]
- early work: Satterthwaite (1959) and Booth and Cox (1962)
- many recent work, e.g., Lin (1993, 1995), Wu (1993), Nguyen (1996), Tang and Wu (1997), Cheng (1997), Li and Wu (1997), etc.
- Most work on 2-level SSDs.

Main question:

- Optimality criteria and Construction

Optimality Criteria

Two-level designs [Booth and Cox (1962)]

- $E(s^2)$: overall average correlation among columns
- $\max(s^2)$: max correlation among columns

Multi-level and mixed-level designs

- $\text{ave}(\chi^2)$ and $\max(\chi^2)$ statistic [Yamada and Lin (1999)]
- $\text{ave}(f)$ and $\max(f)$ [Fang, Lin and Ma (2000)]

Criteria from nonregular designs

- generalized minimum aberration [Xu and Wu (2001)]
  - minimum G_2-aberration [Tang and Deng (1999)]
  - minimum generalized aberration [Ma and Fang (2001)]
- minimum moment aberration [Xu (2003)]

Generalized Minimum Aberration

For a design D of N runs and m factors, consider

$$Y = I\alpha_0 + X_1\alpha_1 + \cdots + X_m\alpha_m + \varepsilon,$$

- Y is the vector of N observations
- α_j is the vector of all j-factor interactions
- X_j is the matrix of orthonormal coefficients for α_j

If $X_j = [x_{ik}^{(j)}]$, let

$$A_j = N^{-2}\|I^T X_j\|^2 = N^{-2}\sum_k \left|\sum_i x_{ik}^{(j)}\right|^2.$$

The GMA criterion (Xu and Wu 2001, *Annals of Statistics*)

- to sequentially minimize $A_1, A_2, A_3, \ldots$.

Example: A 2-Level Design

	X_1			X_2			X_3
	1	2	3	12	13	23	123
1	+	+	+	+	+	+	+
2	+	−	−	−	−	+	+
3	+	+	−	+	−	−	−
4	−	−	+	+	−	−	+
5	−	+	−	−	+	−	+
6	−	−	+	+	−	−	+
Sum	0	0	0	2	−2	−2	4

- $A_1 = (0^2 + 0^2 + 0^2)/6^2 = 0$,
- $A_2 = [2^2 + (-2)^2 + (-2)^2]/6^2 = 1/3$,
- $A_3 = 4^2/6^2 = 4/9$.

Minimum Moment Aberration

For an $N \times m$ matrix $D = [x_{ik}]$, define the tth power moment to be

$$K_t(D) = E_{i<j}\,[\delta_{ij}(D)]^t = [N(N-1)/2]^{-1} \sum_{1 \le i<j \le N} [\delta_{ij}(D)]^t\,,$$

where $\delta_{ij}(D)$ is # of coincidences between the ith and jth rows, i.e., # of k's such that $x_{ik} = x_{jk}$.

- The power moments measure the similarity among the rows.
- A good design should have small power moments.

The MMA criterion (Xu 2003, *Statistica Sinica*)

- to sequentially minimize $K_1, K_2, K_3, \ldots$.

Example: An $OA(27, 13, 3, 2)$

Run	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	1	1	1	1	1	2	2	2	2
3	0	0	0	0	2	2	2	2	2	1	1	1	1
4	0	1	1	2	0	0	1	1	2	0	1	1	2
5	0	1	1	2	1	1	2	2	0	2	0	0	1
6	0	1	1	2	2	2	0	0	1	1	2	2	0
7	0	2	2	1	0	0	2	2	1	0	2	2	1
8	0	2	2	1	1	1	0	0	2	2	1	1	0
9	0	2	2	1	2	2	1	1	0	1	0	0	2
10	1	0	1	1	0	1	0	1	1	1	0	1	1
11	1	0	1	1	1	2	1	2	2	0	2	0	0
12	1	0	1	1	2	0	2	0	0	2	1	2	2
13	1	1	2	0	0	1	1	2	0	1	1	2	0
14	1	1	2	0	1	2	2	0	1	0	0	1	2
15	1	1	2	0	2	0	0	1	2	2	2	0	1
16	1	2	0	2	0	1	2	0	2	1	2	0	2
17	1	2	0	2	1	2	0	1	0	0	1	2	1
18	1	2	0	2	2	0	1	2	1	2	0	1	0
19	2	0	2	2	0	2	0	2	2	2	0	2	2
20	2	0	2	2	1	0	1	0	0	1	2	1	1
21	2	0	2	2	2	1	2	1	1	0	1	0	0
22	2	1	0	1	0	2	1	0	1	2	1	0	1
23	2	1	0	1	1	0	2	1	2	1	0	2	0
24	2	1	0	1	2	1	0	2	0	0	2	1	2
25	2	2	1	0	0	2	2	1	0	2	2	1	0
26	2	2	1	0	1	0	0	2	1	1	1	0	2
27	2	2	1	0	2	1	1	0	2	0	0	2	1

Example: An $OA(27, 13, 3, 2)$ (Cont.)

- The coincidence matrix (δ_{ij}):

Run	1	2	3	4	5	6	$\cdots$	26	27
1	13	4	4	4	4	4	$\cdots$	4	4
2	4	13	4	4	4	4	$\cdots$	4	4
3	4	4	13	4	4	4	$\cdots$	4	4
4	4	4	4	13	4	4	$\cdots$	4	4
5	4	4	4	4	13	4	$\cdots$	4	4
6	4	4	4	4	4	13	$\cdots$	4	4
				$\vdots$					
26	4	4	4	4	4	4	$\cdots$	13	4
27	4	4	4	4	4	4	$\cdots$	4	13

- $K_t = E_{i<j}[(\delta_{i,j})^t]$.

- $K_1 = 4, K_2 = 4^2, K_3 = 4^3$, etc.

Any $OA(27, 13, 3, 2)$ has the same coincidence matrix and moments.

Connection Among Optimality Criteria

For balanced SSDs with N runs and m factors at s levels

- $A_1 = 0$, $K_1 = m(N-s)/[(N-1)s]$.
- $K_2 = [2NA_2 + Nm(m+s-1) - m^2s^2]/[(N-1)s^2]$.
- $E(s^2) = N^2A_2/[m(m-1)/2]$
- $\text{ave}(\chi^2) = NA_2/[m(m-1)/2]$

Min A_2 is equivalent to min $E(s^2)$, $\text{ave}(\chi^2)$ and K_2.

Projected A_2

- A_2 measures the overall aliasing among columns.
- projected A_2 measures the maximum aliasing among columns.

Let $D = (c_1, \ldots, c_m)$.

For any pair of columns (c_i, c_j), define the projected A_2 as

$$A_2(c_i, c_j) = A_2(d), \text{ where } d = (c_i, c_j).$$

- $A_2(D) = \sum_{1 \leq i < j \leq m} A_2(c_i, c_j)$.
- max projected A_2 value is equivalent to $\max(s^2)$ and $\max(\chi^2)$.

Existing Lower Bounds

Suppose D is an $SSD(N, s^m)$.

- A lower bound of $E(s^2)$ [Nguyen (1996); Tang and Wu (1997)]:

$$E(s^2) \geq \frac{N^2(m-N+1)}{(m-1)(N-1)}.$$

- A lower bound of A_2 [Xu (2003)]

$$A_2(D) \geq \frac{m(s-1)(ms-m-N+1)}{2(N-1)}$$

- achieved if and only if $\delta_{ij}(D)$ is constant for all $i < j$.
- achievable when m is a multiple of $(N-1)/(s-1)$.

A New Lower Bound

Theorem 1: Suppose D is an $SSD(N, s^m)$.

$$A_2(D) \geq \frac{m(s-1)(ms-m-N+1)}{2(N-1)} + \frac{(N-1)s^2\eta(1-\eta)}{2N},$$

where η is the fractional part of $\frac{m(N-s)}{(N-1)s}$.

- achieved if and only if $\delta_{ij}(D)$ differs by at most one for all $i < j$.
- An SSD achieving the lower bound is optimal under GMA.
- achievable when $m = q(N-1)/(s-1) + r$ for $r = -1, 0, 1$.
- achievable for any m when $N = s^2$.

Juxtaposition Method

Let $D_1, D_2, \ldots, D_k$ be $OA(27, 13, 3, 2)$.

- Note $\delta_{ij}(D) = 4$ for $i < j$.
- Column juxtaposition of $D_1, D_2, \ldots, D_k$ forms an optimal $SSD(27, 3^{13k})$.
- Removing or adding one column is still optimal.
- extension of Tang and Wu (1997).

May contain fully aliased columns!

Fraction Method

Let D be an $OA(27, 13, 3, 2)$.

- Take any column as the branching column.
- Remove the branching column to obtain 3 one-third fractions D_1, D_2, D_3.
- Each fraction is an optimal $SSD(9, 3^{12})$
- Row juxtaposition of D_1 and D_2 is an optimal $SSD(18, 3^{12})$.
- extension of Lin (1993).

May contain fully aliased columns!

Example: An $OA(27, 13, 3, 2)$

Run	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	1	1	1	1	1	2	2	2	2
3	0	0	0	0	2	2	2	2	2	1	1	1	1
4	0	1	1	2	0	0	1	1	2	0	1	1	2
5	0	1	1	2	1	1	2	2	0	2	0	0	1
6	0	1	1	2	2	2	0	0	1	1	2	2	0
7	0	2	2	1	0	0	2	2	1	0	2	2	1
8	0	2	2	1	1	1	0	0	2	2	1	1	0
9	0	2	2	1	2	2	1	1	0	1	0	0	2
10	1	0	1	1	0	1	0	1	1	1	0	1	1
11	1	0	1	1	1	2	1	2	2	0	2	0	0
12	1	0	1	1	2	0	2	0	0	2	1	2	2
13	1	1	2	0	0	1	1	2	0	1	1	2	0
14	1	1	2	0	1	2	2	0	1	0	0	1	2
15	1	1	2	0	2	0	0	1	2	2	2	0	1
16	1	2	0	2	0	1	2	0	2	1	2	0	2
17	1	2	0	2	1	2	0	1	0	0	1	2	1
18	1	2	0	2	2	0	1	2	1	2	0	1	0
19	2	0	2	2	0	2	0	2	2	2	0	2	2
20	2	0	2	2	1	0	1	0	0	1	2	1	1
21	2	0	2	2	2	1	2	1	1	0	1	0	0
22	2	1	0	1	0	2	1	0	1	2	1	0	1
23	2	1	0	1	1	0	2	1	2	1	0	2	0
24	2	1	0	1	2	1	0	2	0	0	2	1	2
25	2	2	1	0	0	2	2	1	0	2	2	1	0
26	2	2	1	0	1	0	0	2	1	1	1	0	2
27	2	2	1	0	2	1	1	0	2	0	0	2	1

Optimality Results

Theorem 2: Suppose s is a prime power.

(i) An optimal $SSD(s^n, s^m)$ exists for any n and $m = k(s^n - 1)/(s - 1) + r$ where $r = -1, 0, 1$.

(ii) An optimal $SSD(ks^{n-1}, s^m)$ exists for any n, $k < s$ and $m = (s^n - 1)/(s - 1) - 1$.

(iii) An optimal $SSD(s^2, s^m)$ exists for any m.

(iv) An optimal $SSD(ks, s^m)$ exists for any $m \leq s$ and $k < s$.

- All optimal SSD achieve the lower bound in Theorem 1.

Optimality Results (Cont.)

Given N and s, let $a_2(m) = \min\{A_2(D) : D \text{ is an } SSD(N, s^m)\}$.

Theorem 3: Suppose a saturated $OA(N, t, s, 2)$ exists with $t = (N-1)/(s-1)$. Then there exists a positive integer m_0 such that for $m \geq m_0$, $a_2(m+t) = a_2(m) + m(s-1)$.

- A_2 optimal SSDs are periodic when m is large enough.
- Optimal SSDs containing fully aliased columns are not useful in practice.

Question:

- How to construct optimal SSDs without fully aliased columns?

Construction: Idea

- Addelman and Kempthorne (1961) constructed $OA(2s^n, m, s, 2)$ for any prime power s, any n and $m = 2(s^n - 1)/(s - 1) - 1$.
- Such arrays can be naturally decomposed into two arrays.
- Each array is an $SSD(s^n, s^m)$.

Key idea:

- columns are labeled as functions over $F_s = GF(s)$.
- use both linear and quadratic functions.

Example: $OA(18, 7, 3, 2)$

Run	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1
3	0	2	2	2	2	2	2
4	1	0	0	1	1	2	2
5	1	1	1	2	2	0	0
6	1	2	2	0	0	1	1
7	2	0	1	0	2	1	2
8	2	1	2	1	0	2	0
9	2	2	0	2	1	0	1
10	0	0	2	2	1	1	0
11	0	1	0	0	2	2	1
12	0	2	1	1	0	0	2
13	1	0	1	2	0	2	1
14	1	1	2	0	1	0	2
15	1	2	0	1	2	1	0
16	2	0	2	1	2	0	1
17	2	1	0	2	0	1	2
18	2	2	1	0	1	2	0

- Half of the array is an optimal $SSD(9, 3^7)$.

Example: $SSD(9, 3^7)$

- Columns are labeled as linear and quadratic functions.

  $X_1, X_2, X_1+X_2, 2X_1+X_2, X_1^2+X_2, X_1^2+X_1+X_2, X_1^2+2X_1+X_2$

- Evaluating X_1 and X_2 at F_3^2 yields an optimal $SSD(9, 3^7)$

Run	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1
3	0	2	2	2	2	2	2
4	1	0	1	2	1	2	0
5	1	1	2	0	2	0	1
6	1	2	0	1	0	1	2
7	2	0	2	1	1	0	2
8	2	1	0	2	2	1	0
9	2	2	1	0	0	2	1

- Note $\delta_{ij} = 1$ for $i < j$.

- Any projection design is also optimal.

Construction: Notation

Let $X_1, \ldots, X_n$ be n variables over $F_s = GF(s)$.

$$\begin{aligned} H(X_1, \ldots, X_n) &= \{c_1 X_1 + \cdots + c_n X_n : c_i \in F_s, \\ &\quad \text{not all } c_i \text{ are 0 and the } \textbf{last} \text{ nonzero } c_i \text{ is } 1\} \\ Q_1^*(X_1, \ldots, X_n) &= \{X_1^2 + aX_1 + h : a \in F_s, h \in H(X_2, \ldots, X_n)\} \\ Q_1(X_1, \ldots, X_n) &= \{X_1\} \cup Q_1^*(X_1, \ldots, X_n). \end{aligned}$$

- Both $H(X_1, \ldots, X_n)$ and $Q_1(X_1, \ldots, X_n)$ are saturated $OA(s^n, (s^n - 1)/(s - 1), s, 2)$ when evaluated at F_s^n.
- $H(X_1, \ldots, X_n)$ is a regular fractional factorial design.
- $Q_1(X_1, \ldots, X_n)$ is isomorphic to $H(X_1, \ldots, X_n)$ when $n = 2$.
- $Q_1(X_1, \ldots, X_n)$ is NOT isomorphic to $H(X_1, \ldots, X_n)$ when $n > 2$ and $s > 2$.

Main Results

Theorem 4: Column juxtaposition of $H(X_1, \ldots, X_n)$ and $Q_1^*(X_1, \ldots, X_n)$ is

- an optimal $SSD(s^n, s^m)$ with $m = 2(s^n - 1)/(s - 1) - 1$.
- an overall $A_2 = s^n - s$.
- X_1 is orthogonal to all other columns.
- no fully aliased columns if $s > 2$.
- $s(s^n - s)/(s - 1)$ pairs of non-orthogonal columns with projected $A_2 = (s - 1)/s$ for s odd.
- $s^n - s$ pairs of non-orthogonal columns with projected $A_2 = 1$ for s even.

Main Results (Cont.)

Can construct $Q_h(X_1, \ldots, X_n)$ for each $h \in H(X_1, \ldots, X_n)$.

Theorem 6: For $k \le (s^n - 1)/(s - 1)$, column juxtaposition of k $Q_h(X_1, \ldots, X_n)$ is

- an optimal $SSD(s^n, s^m)$ with $m = k(s^n - 1)/(s - 1)$.
- overall $A_2 = \binom{k}{2}(s^n - 1)$.
- no fully aliased columns if s is odd or $s > 4$.
- For s odd, $\binom{k}{2}2s$ pairs with projected $A_2 = (s - 1)/s$, $\binom{k}{2}s^2$ pairs with projected $A_2 = (s - 1)^2/s^2$, and $\binom{k}{2}s^2(s^n - s^2)/(s - 1)$ pairs with projected $A_2 = (s - 1)/s^2$.
- For s even, possible projected A_2 values are $0, 1, 2$ and 3.
- For $s = 4$, $\binom{k}{2}$ pairs with projected $A_2 = 3$ and $\binom{k}{2}(4^n - 4)$ pairs with projected $A_2 = 1$.

Main Results (Cont.)

All quadratic functions form another class of SSDs, which have projected $A_2 = (s-1)/s$ for s odd.

Theorem 7: Suppose s is odd. For $k \leq (s^n-1)/(s-1)$, column juxtaposition of all quadratic functions in k $Q_h(X_1, \ldots, X_n)$ is

- an $SSD(s^n, s^m)$ with $m = k(s^n - s)/(s-1)$.
- overall $A_2 = \binom{k}{2}(s^n - 2s + 1)$.
- optimal when $k = (s^n-1)/(s-1) - 1$ or $(s^n-1)/(s-1)$.
- no fully aliased columns.
- $\binom{k}{2}s^2$ pairs with projected $A_2 = (s-1)^2/s^2$, and $\binom{k}{2}s^2(s^n - s^2)/(s-1)$ pairs with projected $A_2 = (s-1)/s^2$.

Example: $s = 3$ **and** $n = 2$

Note $H(X_1, X_2) = \{X_1, X_2, X_1 + X_2, 2X_1 + X_2\}$.

$$
\begin{aligned}
Q_{X_1}(X_1, X_2) &= \{X_1, X_1^2 + X_2, X_1^2 + X_1 + X_2, X_1^2 + 2X_1 + X_2\}, \\
Q_{X_2}(X_1, X_2) &= \{X_2, X_2^2 + X_1, X_2^2 + X_2 + X_1, X_2^2 + 2X_2 + X_1\}, \\
Q_{X_1+X_2}(X_1, X_2) &= \{X_1 + X_2, (X_1 + X_2)^2 + X_1, \\
&\quad (X_1 + X_2)^2 + 2X_1 + X_2, (X_1 + X_2)^2 + 2X_2\}, \\
Q_{2X_1+X_2}(X_1, X_2) &= \{2X_1 + X_2, (2X_1 + X_2)^2 + X_1, \\
&\quad (2X_1 + X_2)^2 + X_2, (2X_1 + X_2)^2 + 2X_1 + 2X_2\}.
\end{aligned}
$$

- The 16 columns together form an optimal $SSD(9, 3^{16})$ with an overall $A_2 = 48$ and max projected $A_2 = 2/3$.
- The 12 quadratic columns together form an optimal $SSD(9, 3^{12})$ with an overall $A_2 = 24$ and max projected $A_2 = 4/9$.

Main Results (Cont.)

Fractions of $H(X_1, \ldots, X_n)$:

Theorem 8: k/s fractions with any branching column

- an optimal $SSD(ks^{n-1}, s^m)$ with $m = (s^n - s)/(s-1)$.
- overall $A_2 = (s^n - s)(s-k)/(2k)$.
- no fully aliased columns for $1 < k < s$.
- $(s^n - s)/2$ pairs of nonorthogonal columns with projected $A_2 = (s-k)/k$.

Main Results (Cont.)

Fractions of $Q_1^*(X_1, \ldots, X_n)$:

Theorem 9: k/s fractions with branching column $X_1^2 + X_2$

- an optimal $SSD(ks^{n-1}, s^m)$ with $m = (s^n - s)/(s-1)$.
- overall $A_2 = (s^n - s)(s-k)/(2k)$.
- no fully aliased columns for $1 < k < s$.
- max projected $A_2 = (s-k)/k$ for s odd.
- max projected $A_2 \leq \max\{(s-k)/k, 1\}$ for s even.
- exact frequency for projected A_2 for s odd or $s = 4$.

Some Optimal 3-Level SSDs

		Projected A_2 Values					
N	m	1/6	2/9	4/9	1/2	2/3	Source
6	3				3		Th. 8, $n = 2, k = 2$
9	7					9	Th. 4, $n = 2$
9	12			54			Th. 7, $n = 2, k = 4$
9	16			54		36	Th. 6, $n = 2, k = 4$
18	12				12		Th. 8, $n = 3, k = 2$
18	12	27			3		Th. 9, $n = 3, k = 2$
27	25					36	Th. 4, $n = 3$
27	26		81	9		6	Th. 6, $n = 3, k = 2$
27	156		6318	702			Th. 7, $n = 3, k = 13$
27	169		6318	702		468	Th. 6, $n = 3, k = 13$
54	39				39		Th. 8, $n = 4, k = 2$
54	39	108			3		Th. 9, $n = 4, k = 2$

Some Optimal 4-Level SSDs

		Projected A_2 Values			
N	m	1/9	1/3	1	Source
8	4			6	Th. 8, $n = 2, k = 2$
12	4		6		Th. 8, $n = 2, k = 3$
16	9			12	Th. 4, $n = 2$
16	15			45	Th. 6[a], $n = 2, k = 5$
32	20			30	Th. 8, $n = 3, k = 2$
48	20		30		Th. 8, $n = 3, k = 3$
48	20	72	6		Th. 9, $n = 3, k = 3$
64	41			60	Th. 4, $n = 3$
64	231			3465	Th. 6[a], $n = 3, k = 21$

[a] The design is obtained by removing fully aliased columns.

Some Optimal 5-Level SSDs

		Projected A_2 Values							
N	m	$\frac{2}{15}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{16}{25}$	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{3}{2}$	Source
10	5							10	Th. 8, $n = 2, k = 2$
15	5					10			Th. 8, $n = 2, k = 3$
20	5		10						Th. 8, $n = 2, k = 4$
25	11						25		Th. 4, $n = 2$
25	30				375				Th. 7, $n = 2, k = 6$
25	36				375		150		Th. 6, $n = 2, k = 6$
50	30							60	Th. 8, $n = 3, k = 2$
50	30			250				10	Th. 9, $n = 3, k = 2$
75	30					60			Th. 8, $n = 3, k = 3$
75	30	250				10			Th. 9, $n = 3, k = 3$

Comparison in terms of ave(f)

N	s	m	Th. 6	Th. 7	FLM	LS	AG
9	3	8	2.57	3.00	2.57	2.57	2.43
9	3	12	3.27	3.27	3.27	3.27	3.06
9	3	16	3.60		3.60	3.60	
9	3	28				4.00	
16	4	10	6.04		4.36	4.84	4.93
16	4	15	6.86		5.60	6.23	6.27
16	4	20			6.25	6.95	
16	4	40				7.87	
25	5	12	8.33	9.55	6.42	8.06	7.45
25	5	18	10.78	10.98	8.41	10.42	9.52
25	5	24	11.96	11.67	10.20	11.66	10.86
25	5	30	12.64	12.07		12.33	11.33
25	5	36	13.10			12.73	
27	3	26	3.66	3.77	3.78		4.26
27	3	39	4.81	4.81	5.27		5.63
27	3	52	5.38	5.97	5.98		6.32
27	3	65	5.71	6.28			6.73
27	3	156	6.49	6.97			
27	3	169	6.53				

Fang et al, (2000); Lu and Sun (2001); Aggarwal and Gupta (2004).

Comparison in terms of $\max(f)$

N	s	m	Th. 6	Th. 7	FLM	LS	AG
9	3	8	6	4	6	6	8
9	3	12	6	4	6	6	8
9	3	16	6		6	6	
9	3	28				6	
16	4	10	16		12	12	12
16	4	15	16		12	12	14
16	4	20			16	12	
16	4	40				16	
25	5	12	20	14	22	18	24
25	5	18	20	14	24	20	24
25	5	24	20	14	30	20	32
25	5	30	20	14		22	32
25	5	36	20			22	
27	3	26	18	12	16		16
27	3	39	18	12	18		16
27	3	52	18	12	18		16
27	3	65	18	12			16
27	3	156	18	12			
27	3	169	18				

Fang et al, (2000); Lu and Sun (2001); Aggarwal and Gupta (2004).

Summary of Comparisons

- Our SSDs are competitive in terms of max(f) but less competitive in terms of ave(f).
- For $N = 9, 25, 27$, SSDs based on Theorem 7 are better than existing ones in terms of both max(χ^2) and max(f).
- In terms of ave(f), our SSDs are worse than existing ones for $N = 9, 25$ but better for $N = 27$.
- For $N = 16$, our SSDs are less competitive.

Advantages of algebraic methods over algorithmic methods:

- not limited to small run sizes.
- more efficient in control of max aliasing among columns (for large N).

Mixed-Level Designs

GMA criterion works for mixed-level SSDs.

Theorem 9: For an $SSD(N, s_1 s_2 \cdots s_m)$,

$$A_2 \geq \frac{(\sum s_k - m)(\sum s_k - m - N + 1)}{2(N-1)}.$$

- is equivalent to the lower bound of ave(χ^2) [Yamada and Matsui (2002)]
- is equivalent to the lower bound of K_2 [Xu (2003)].

Mixed-Level Designs: Construction

Method of Replacement

- An $SSD(81, 9^{100})$ ($n = 2$, $s = 9$ and $k = 10$ in Th. 6)
  - overall $A_2 = 3600$ and maximum projected $A_2 = 8/9$.
- Replace a 9-level column with an $OA(9, 4, 3, 2)$.
- $SSD(81, 9^{100-i}3^{4i})$ for $1 \leq i < 100$
  - overall $A_2 = 3600$ and projected $A_2 \leq 8/9$.
  - achieve the lower bound of A_2 in Theorem 9.

Summary

- General optimality results
- Explicit algebraic construction of optimal SSDs.
  - Half Addelman-Kempthorne arrays
  - Juxtaposition of saturated OAs
  - Fractions of saturated OAs
- Design properties are studied analytically.

Tables of SSDs and manuscript are available at

http://www.stat.ucla.edu/~hqxu/