7 Sampling Distributions and Point Estimation of Parameters

Cramer-Rao Inequality
Fisher Information

CHAPTER OUTLINE

7-1 Introduction

- The field of statistical inference consists of those methods used to make decisions or to draw conclusions about a **population**.

- These methods utilize the information contained in a **sample** from the population in drawing conclusions.

- Statistical inference may be divided into two major areas:

    - Parameter estimation

    - Hypothesis testing

7-1 Introduction

Suppose that we want to obtain a point estimate of a population parameter. We know that before the data is collected, the observations are considered to be random variables, say $X_1, X_2, \ldots, X_n$. Therefore, any function of the observation, or any **statistic,** is also a random variable. For example, the sample mean $\overline{X}$ and the sample variance S^2 are statistics and they are also random variables.

Since a statistic is a random variable, it has a probability distribution. We call the probability distribution of a statistic a **sampling distribution.** The notion of a sampling distribution is very important and will be discussed and illustrated later in the chapter.

Definition

A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$. The statistic $\hat{\Theta}$ is called the **point estimator**.

7-1 Introduction

Estimation problems occur frequently in engineering. We often need to estimate

- The mean μ of a single population
- The variance σ^2 (or standard deviation σ) of a single population
- The proportion p of items in a population that belong to a class of interest
- The difference in means of two populations, $\mu_1 - \mu_2$
- The difference in two population proportions, $p_1 - p_2$

7-1 Introduction

Reasonable point estimates of these parameters are as follows:

- For μ, the estimate is $\hat{\mu} = \bar{x}$, the sample mean.
- For σ^2, the estimate is $\hat{\sigma}^2 = s^2$, the sample variance.
- For p, the estimate is $\hat{p} = x/n$, the sample proportion, where x is the number of items in a random sample of size n that belong to the class of interest.
- For $\mu_1 - \mu_2$, the estimate is $\hat{\mu}_1 - \hat{\mu}_2 = \bar{x}_1 - \bar{x}_2$, the difference between the sample means of two independent random samples.
- For $p_1 - p_2$, the estimate is $\hat{p}_1 - \hat{p}_2$, the difference between two sample proportions computed from two independent random samples.

7.2 Sampling Distributions and the Central Limit Theorem

Statistical inference is concerned with making decisions about a population based on the information contained in a random sample from that population.

Definitions:

The random variables $X_1, X_2, \ldots, X_n$ are a **random sample** of size n if (a) the X_i's are independent random variables, and (b) every X_i has the same probability distribution.

A **statistic** is any function of the observations in a random sample.

The probability distribution of a statistic is called a **sampling distribution.**

7.2 Sampling Distributions

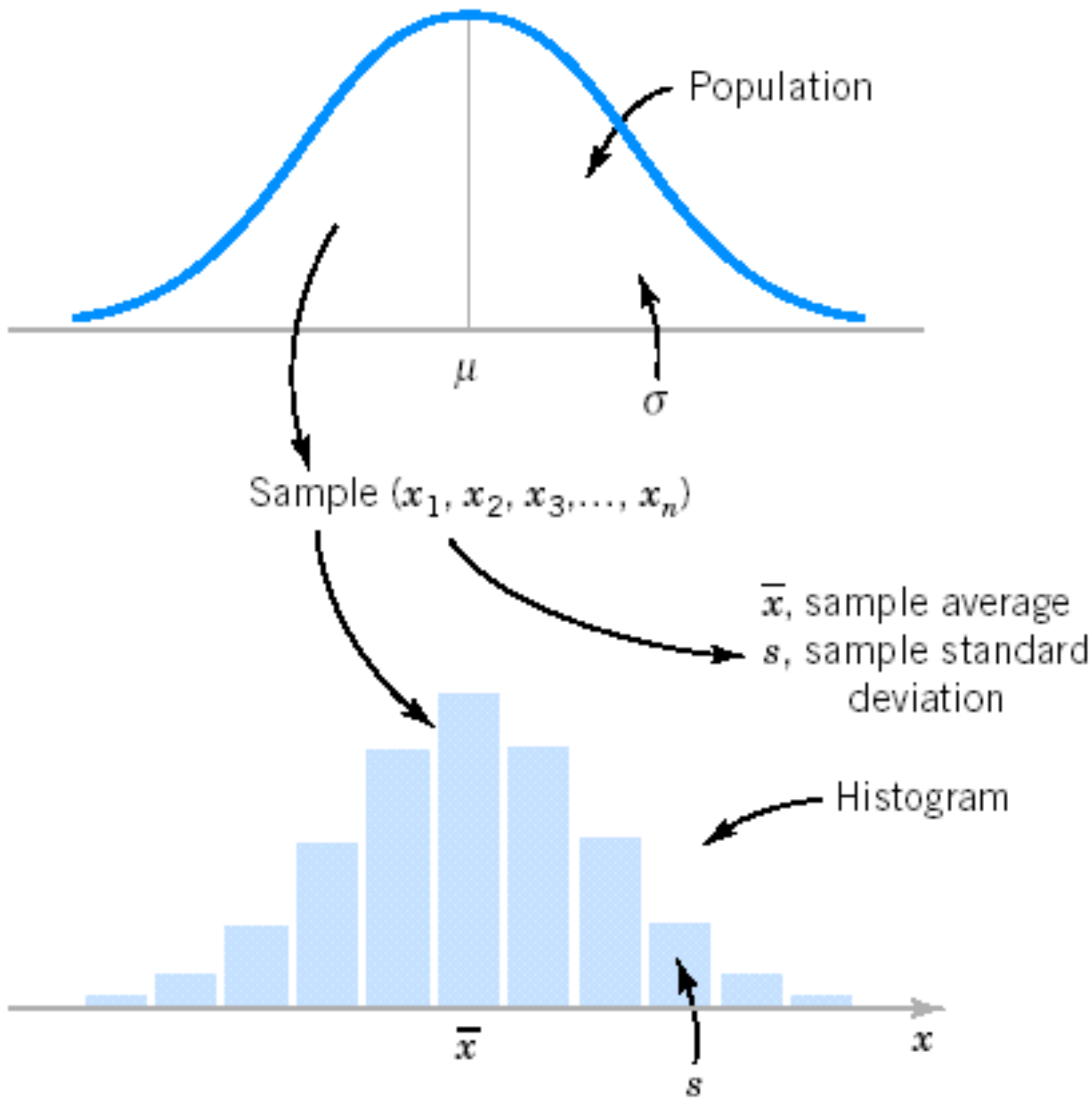


Figure 6-3
Relationship between a population and a sample.

7.2 Sampling Distributions

Suppose $X_1, \ldots, X_n$ are a random sample from a population with mean μ and variance σ^2.

(a) What are the mean and variance of the sample mean?

(b) What is the sampling distribution of the sample mean if the population is normal.

7.2 Sampling Distributions and the Central Limit Theorem

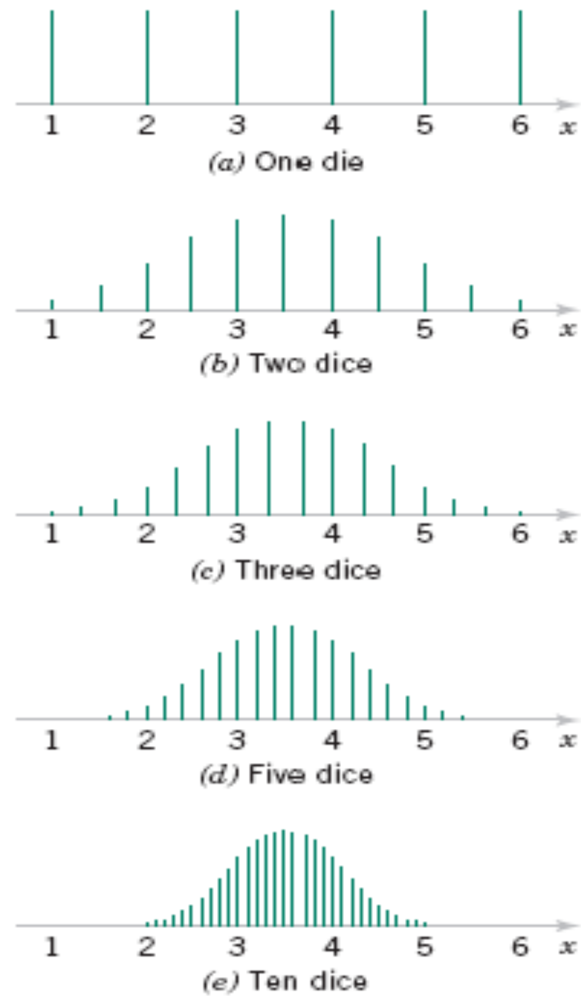
If we are sampling from a population that has an unknown probability distribution, the sampling distribution of the sample mean will still be approximately normal with mean μ and variance σ^2/n, if the sample size n is large. This is one of the most useful theorems in statistics, called the **central limit theorem.** The statement is as follows:

> If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population (either finite or infinite) with mean μ and finite variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of
>
> $$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{7-1}$$
>
> as $n \rightarrow \infty$, is the standard normal distribution.

If the population is normal, the sampling distribution of Z is exactly standard normal.

7.2 Sampling Distributions and the Central Limit Theorem

Figure 7-1 Distributions of average scores from throwing dice. [Adapted with permission from Box, Hunter, and Hunter (1978).]



CLT Simulation

7.2 Sampling Distributions and the Central Limit Theorem

Example 7-1

An electronics company manufactures resistors that have a mean resistance of 100 ohms and a standard deviation of 10 ohms. The distribution of resistance is normal. Find the probability that a random sample of $n = 25$ resistors will have an average resistance less than 95 ohms.

Note that the sampling distribution of $\overline{X}$ is normal, with mean $\mu_{\overline{X}} = 100$ ohms and a standard deviation of

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$$

Therefore, the desired probability corresponds to the shaded area in Fig. 7-1. Standardizing the point $\overline{X} = 95$ in Fig. 7-2, we find that

$$z = \frac{95 - 100}{2} = -2.5$$

and therefore,

$$P(\overline{X} < 95) = P(Z < -2.5)$$
$$= 0.0062$$

7.2 Sampling Distributions and the Central Limit Theorem

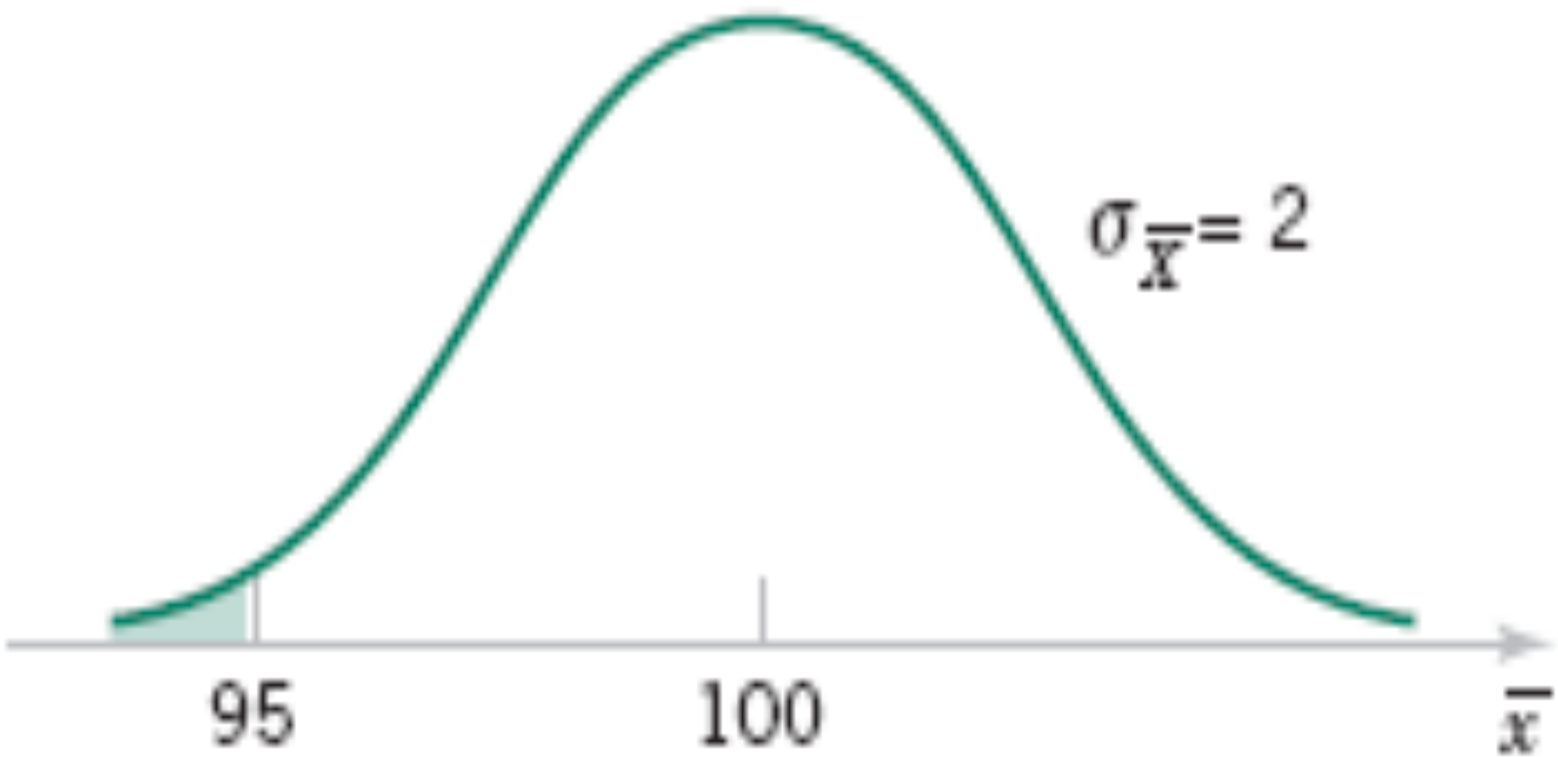


Figure 7-2 Probability for Example 7-1

7.2 Sampling Distributions and the Central Limit Theorem

Approximate Sampling Distribution of a Difference in Sample Means

If we have two independent populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 and if $\overline{X}_1$ and $\overline{X}_2$ are the sample means of two independent random samples of sizes n_1 and n_2 from these populations, then the sampling distribution of

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \tag{7-4}$$

is approximately standard normal, if the conditions of the central limit theorem apply. If the two populations are normal, the sampling distribution of Z is exactly standard normal.

7-3 General Concepts of Point Estimation

7-3.1 Unbiased Estimators

Definition

The point estimator $\hat{\Theta}$ is an **unbiased estimator** for the parameter θ if

$$E(\hat{\Theta}) = \theta \qquad (7\text{-}5)$$

If the estimator is not unbiased, then the difference

$$E(\hat{\Theta}) - \theta \qquad (7\text{-}6)$$

is called the **bias** of the estimator $\hat{\Theta}$.

7-3 General Concepts of Point Estimation

Example 7-4

Suppose that X is a random variable with mean μ and variance σ^2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the population represented by X. Show that the sample mean $\overline{X}$ and sample variance S^2 are unbiased estimators of μ and σ^2, respectively.

First consider the sample mean. In Equation 5.40a in Chapter 5, we showed that $E(\overline{X}) = \mu$. Therefore, the sample mean $\overline{X}$ is an unbiased estimator of the population mean μ.

Now consider the sample variance. We have

$$E(S^2) = E\left[\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}\right] = \frac{1}{n-1} E \sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$= \frac{1}{n-1} E \sum_{i=1}^{n}(X_i^2 + \overline{X}^2 - 2\overline{X}X_i) = \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} E(X_i^2) - nE(\overline{X}^2)\right]$$

7-3 General Concepts of Point Estimation

Example 7-4 (continued)

The last equality follows from Equation 5-37 in Chapter 5. However, since $E(X_i^2) = \mu^2 + \sigma^2$ and $E(\overline{X}^2) = \mu^2 + \sigma^2/n$, we have

$$\begin{aligned} E(S^2) &= \frac{1}{n-1}\left[\sum_{i=1}^{n} (\mu^2 + \sigma^2) - n(\mu^2 + \sigma^2/n)\right] \\ &= \frac{1}{n-1}(n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2) \\ &= \sigma^2 \end{aligned}$$

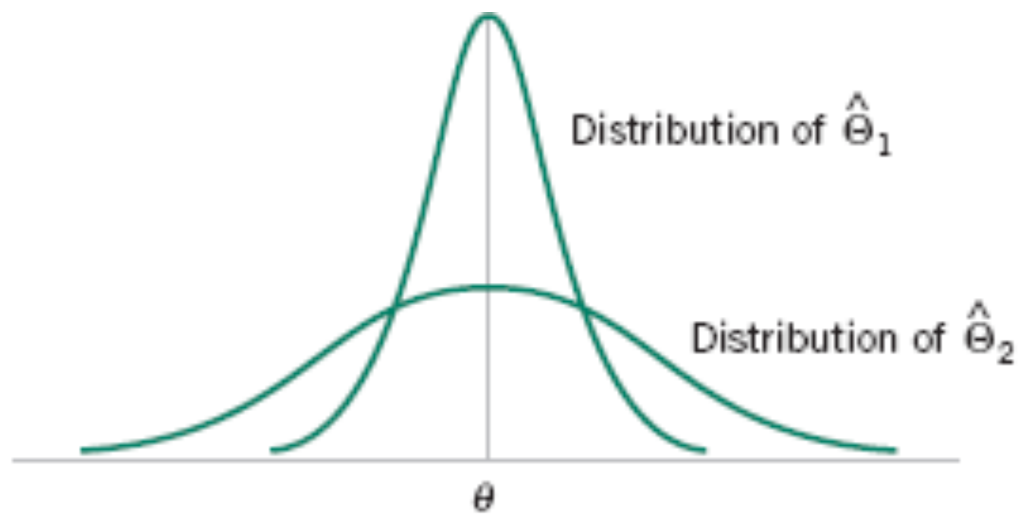
Therefore, the sample variance S^2 is an unbiased estimator of the population variance σ^2.

7-3.2 Variance of a Point Estimator

If we consider all unbiased estimators of θ, the one with the smallest variance is called the **minimum variance unbiased estimator** (MVUE).

Figure 7-5 The sampling distributions of two unbiased estimators

$\hat{\Theta}_1$ and $\hat{\Theta}_2$.



If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a normal distribution with mean μ and variance σ^2, the sample mean $\overline{X}$ is the MVUE for μ.

7-3.3 Standard Error: Reporting a Point Estimate

The **standard error** of an estimator $\hat{\Theta}$ is its standard deviation, given by $\sigma_{\hat{\Theta}} = \sqrt{V(\hat{\Theta})}$. If the standard error involves unknown parameters that can be estimated, substitution of those values into $\sigma_{\hat{\Theta}}$ produces an **estimated standard error,** denoted by $\hat{\sigma}_{\hat{\Theta}}$.

Suppose we are sampling from a normal distribution with mean μ and variance σ^2. Now the distribution of $\overline{X}$ is normal with mean μ and variance σ^2/n, so the standard error of $\overline{X}$ is

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

If we did not know σ but substituted the sample standard deviation S into the above equation, the estimated standard error of $\overline{X}$ would be

$$\hat{\sigma}_{\overline{X}} = \frac{S}{\sqrt{n}}$$

7-3.3 Standard Error: Reporting a Point Estimate

Example 7-5

An article in the *Journal of Heat Transfer* (Trans. ASME, Sec. C, 96, 1974, p. 59) described a new method of measuring the thermal conductivity of Armco iron. Using a temperature of 100°F and a power input of 550 watts, the following 10 measurements of thermal conductivity (in Btu/hr-ft-°F) were obtained:

41.60, 41.48, 42.34, 41.95, 41.86,
42.18, 41.72, 42.26, 41.81, 42.04

A point estimate of the mean thermal conductivity at 100°F and 550 watts is the sample mean or

$$\bar{x} = 41.924 \text{ Btu/hr-ft-°F}$$

7-3.3 Standard Error: Reporting a Point Estimate

Example 7-5 (continued)

The standard error of the sample mean is $\sigma_{\overline{X}} = \sigma/\sqrt{n}$, and since σ is unknown, we may replace it by the sample standard deviation $s = 0.284$ to obtain the estimated standard error of $\overline{X}$ as

$$\hat{\sigma}_{\overline{X}} = \frac{s}{\sqrt{n}} = \frac{0.284}{\sqrt{10}} = 0.0898$$

Notice that the standard error is about 0.2 percent of the sample mean, implying that we have obtained a relatively precise point estimate of thermal conductivity. If we can assume that thermal conductivity is normally distributed, 2 times the standard error is $2\hat{\sigma}_{\overline{X}} = 2(0.0898) = 0.1796$, and we are highly confident that the true mean thermal conductivity is with the interval 41.924 ± 0.1756, or between 41.744 and 42.104.

7-3.4 Mean Square Error of an Estimator

The **mean squared error** of an estimator $\hat{\Theta}$ of the parameter θ is defined as

$$\text{MSE}(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 \qquad (7\text{-}7)$$

The mean squared error is an important criterion for comparing two estimators. Let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be two estimators of the parameter θ, and let MSE $(\hat{\Theta}_1)$ and MSE $(\hat{\Theta}_2)$ be the mean squared errors of $\hat{\Theta}_1$ and $\hat{\Theta}_2$. Then the **relative efficiency** of $\hat{\Theta}_2$ to $\hat{\Theta}_1$ is defined as

$$\frac{\text{MSE}(\hat{\Theta}_1)}{\text{MSE}(\hat{\Theta}_2)} \qquad (7\text{-}8)$$

If this relative efficiency is less than 1, we would conclude that $\hat{\Theta}_1$ is a more efficient estimator of θ than $\hat{\Theta}_2$, in the sense that it has a smaller mean square error.

7-3.4 Mean Square Error of an Estimator

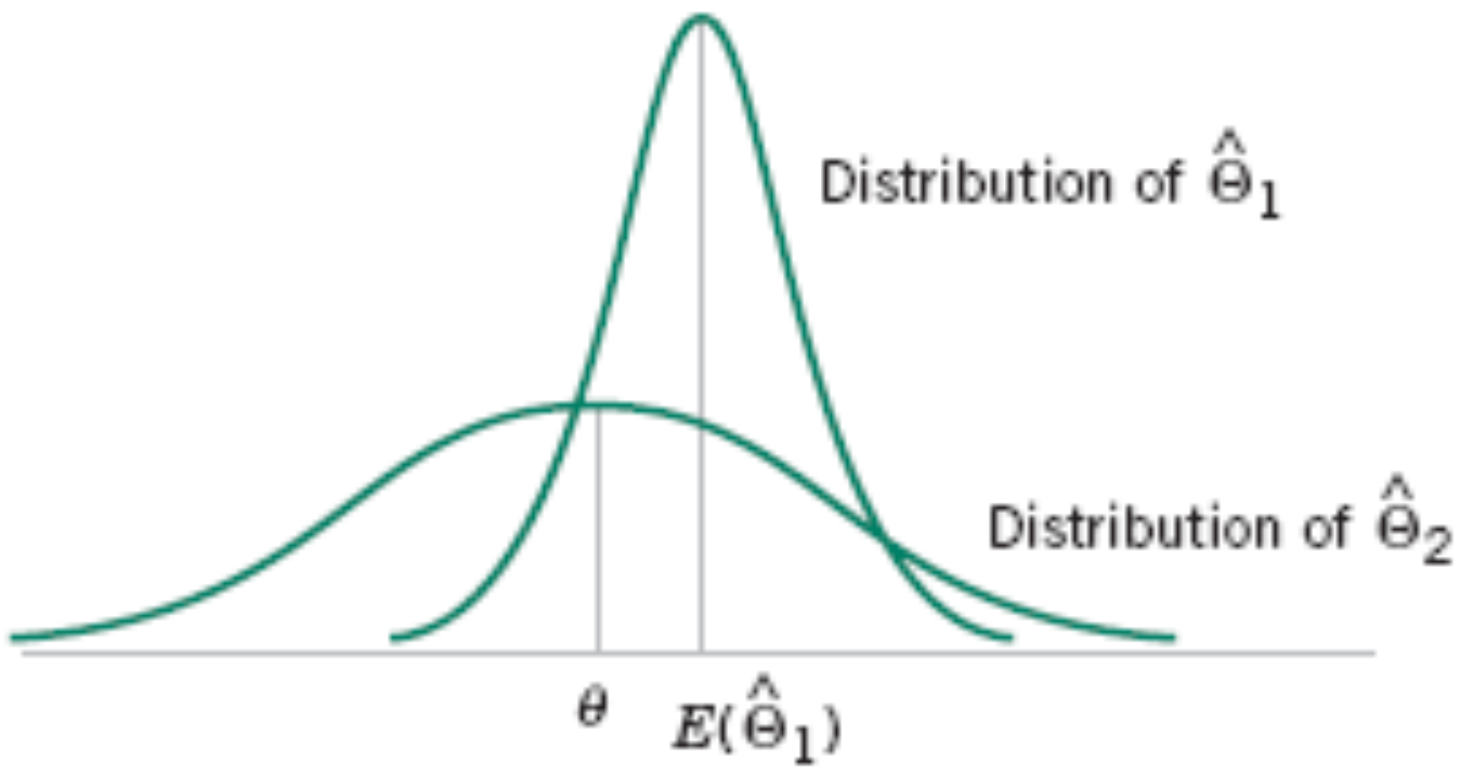


Figure 7-6 A biased estimator $\hat{\Theta}_1$ that has smaller variance than the unbiased estimator $\hat{\Theta}_2$.

7-4 Methods of Point Estimation

- Problem: To find p=P(heads) for a biased coin.
- Procedure: Flip the coin n times.
- Data (a random sample) : $X_1, X_2, \ldots, X_n$
  - where X_i=1 or 0 if the ith outcome is heads or tails.
- Question: How to estimate p using the data?

7-4 Methods of Point Estimation

Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample from the probability distribution $f(x)$, where $f(x)$ can be a discrete probability mass function or a continuous probability density function. The kth **population moment** (or **distribution moment**) is $E(X^k)$, $k = 1, 2, \ldots$. The corresponding kth **sample moment** is $(1/n)\sum_{i=1}^{n} X_i^k$, $k = 1, 2, \ldots$.

Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample from either a probability mass function or probability density function with m unknown parameters $\theta_1, \theta_2, \ldots, \theta_m$. The **moment estimators** $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_m$ are found by equating the first m population moments to the first m sample moments and solving the resulting equations for the unknown parameters.

7-4 Methods of Point Estimation

Example 7-7: Consider normal distribution $N(\mu,\sigma^2)$.

Find the moment estimators of μ and σ^2.

7-4 Methods of Point Estimation

7-4.2 Method of Maximum Likelihood

Definition

Suppose that X is a random variable with probability distribution $f(x; \theta)$, where θ is a single unknown parameter. Let $x_1, x_2, \ldots, x_n$ be the observed values in a random sample of size n. Then the **likelihood function** of the sample is

$$L(\theta) = f(x_1; \theta) \cdot f(x_2; \theta) \cdot \cdots \cdot f(x_n; \theta) \qquad (7\text{-}9)$$

Note that the likelihood function is now a function of only the unknown parameter θ. The **maximum likelihood estimator (MLE)** of θ is the value of θ that maximizes the likelihood function $L(\theta)$.

7-4 Methods of Point Estimation

Example 7-9

Let X be a Bernoulli random variable. The probability mass function is

$$f(x;p) = \begin{cases} p^x(1-p)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

where p is the parameter to be estimated. The likelihood function of a random sample of size n is

$$\begin{aligned} L(p) &= p^{x_1}(1-p)^{1-x_1} p^{x_2}(1-p)^{1-x_2} \cdots p^{x_n}(1-p)^{1-x_n} \\ &= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i} \end{aligned}$$

7-4 Methods of Point Estimation

Example 7-9 (continued)

We observe that if $\hat{p}$ maximizes $L(p)$, $\hat{p}$ also maximizes $\ln L(p)$. Therefore,

$$\ln L(p) = \left(\sum_{i=1}^{n} x_i\right) \ln p + \left(n - \sum_{i=1}^{n} x_i\right) \ln(1 - p)$$

Now

$$\frac{d \ln L(p)}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\left(n - \sum_{i=1}^{n} x_i\right)}{1 - p}$$

Equating this to zero and solving for p yields $\hat{p} = (1/n) \sum_{i=1}^{n} x_i$. Therefore, the maximum likelihood estimator of p is

$$\hat{P} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

7-4 Methods of Point Estimation

Examples 7-6 and 7-11

The time to failure of an electronic module used in an automobile engine controller is tested at an elevated temperature to accelerate the failure mechanism. The time to failure is **exponentially distributed**. Eight units are randomly selected and tested, resulting in the following failure time (in hours): 11.96, 5.03, 67.40, 16.07, 31.50, 7.73, 11.10, 22.38.

Here X is exponentially distributed with parameter λ.

(a)What is the moment estimate of λ?

(b) What is the MLE estimate of λ?

7-4 Methods of Point Estimation

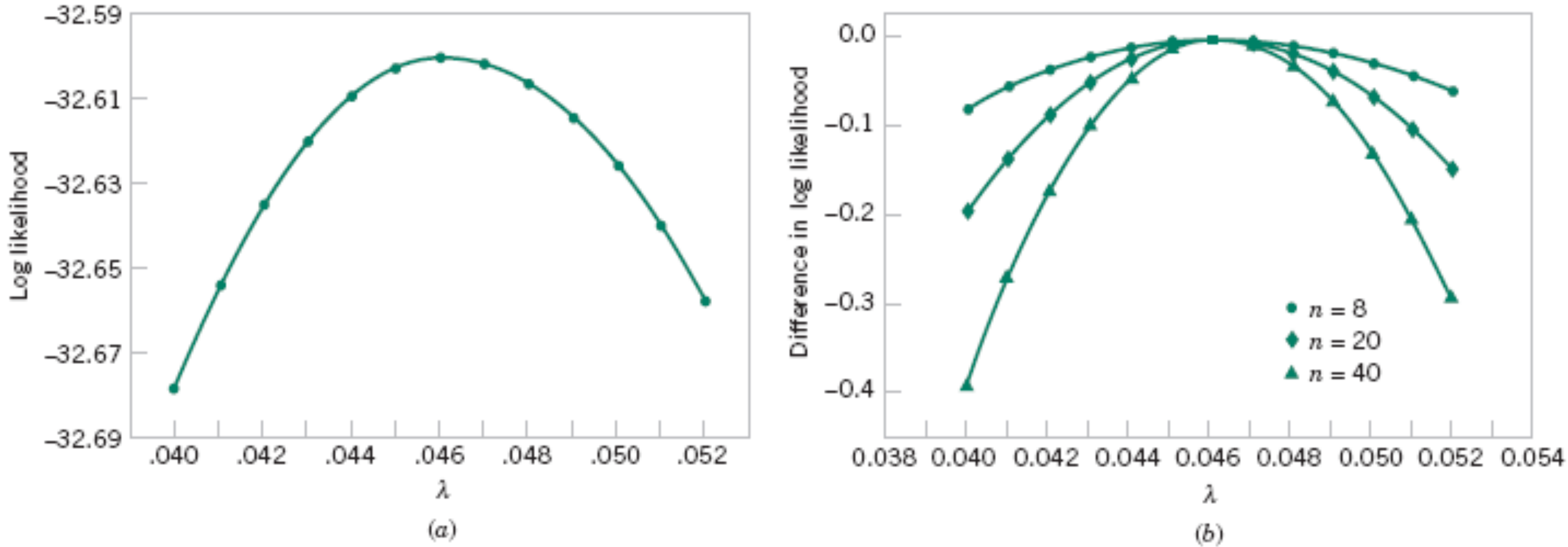


Figure 7-7 Log likelihood for the exponential distribution, using the failure time data. (a) Log likelihood with n = 8 (original data). (b) Difference in Log likelihood if n = 8, 20, and 40.

7-4 Methods of Point Estimation

Example 7-12

Let X be normally distributed with mean μ and variance σ^2, where both μ and σ^2 are unknown. The likelihood function for a random sample of size n is

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_i-\mu)^2/(2\sigma^2)} = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)\sum_{i=1}^{n}(x_i-\mu)^2}$$

and

$$\ln L(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

7-4 Methods of Point Estimation

Example 7-12 (continued)

Now

$$\frac{\partial \ln L(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) = 0$$

$$\frac{\partial \ln L(\mu, \sigma^2)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

The solutions to the above equation yield the maximum likelihood estimators

$$\hat{\mu} = \overline{X} \qquad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Once again, the maximum likelihood estimators are equal to the moment estimators.

7-4 Methods of Point Estimation

Cramer-Rao Inequality (extra!)

Let $X_1, X_2, \cdots, X_n$ be a random sample with pdf $f(x, \theta)$.

If $\hat{\Theta}$ is an unbiased estimator of θ, then

$$\text{var}(\hat{\Theta}) \geq \frac{1}{nI(\theta)}$$

where

$$I(\theta) = E\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right]^2 = -E\left[\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right]$$

is the Fisher information.

7-4 Methods of Point Estimation

Properties of the Maximum Likelihood Estimator

Under very general and not restrictive conditions, when the sample size n is large and if $\hat{\Theta}$ is the maximum likelihood estimator of the parameter θ,

(1) $\hat{\Theta}$ is an approximately unbiased estimator for θ $[E(\hat{\Theta}) \simeq \theta]$,

(2) the variance of $\hat{\Theta}$ is nearly as small as the variance that could be obtained with any other estimator, and

(3) $\hat{\Theta}$ has an approximate normal distribution.

7-4 Methods of Point Estimation

The Invariance Property

Let $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k$ be the maximum likelihood estimators of the parameters $\theta_1, \theta_2, \ldots, \theta_k$. Then the maximum likelihood estimator of any function $h(\theta_1, \theta_2, \ldots, \theta_k)$ of these parameters is the same function $h(\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k)$ of the estimators $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k$.

In the normal distribution case, the maximum likelihood estimators of μ and σ^2 were $\hat{\mu} = \overline{X}$ and $\hat{\sigma}^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$. To obtain the maximum likelihood estimator of the function $h(\mu, \sigma^2) = \sqrt{\sigma^2} = \sigma$, substitute the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ into the function h, which yields

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^{1/2}$$

Thus, the maximum likelihood estimator of the standard deviation σ is *not* the sample standard deviation S.

7-4 Methods of Point Estimation

Complications in Using Maximum Likelihood Estimation

- It is not always easy to maximize the likelihood function because the equation(s) obtained from $dL(\theta)/d\theta = 0$ may be difficult to solve.
- It may not always be possible to use calculus methods directly to determine the maximum of $L(\theta)$.
- See Example 7-14.