10

Statistical Inference for Two Samples

CHAPTER OUTLINE

# 10-1 Introduction

The previous chapter presented hypothesis tests and confidence intervals for a single population parameter (the mean μ, the variance σ^2, or a proportion p). This chapter extends those results to the case of two independent populations.

The general situation is shown in Fig. 10-1. Population 1 has mean μ_1 and variance σ_1^2, while population 2 has mean μ_2 and variance σ_2^2. Inferences will be based on two random samples of sizes n1 and n2, respectively. That is, X11, X12, p , X_{1n_1} is a random sample of n1 observations from population 1, and X21, X22, p , X_{2n_2} is a random sample of n2 observations from population 2. Most of the practical applications of the procedures in this chapter arise in the context of **simple comparative experiments** in which the objective is to study the difference in the parameters of the two populations.

10-1 Introduction
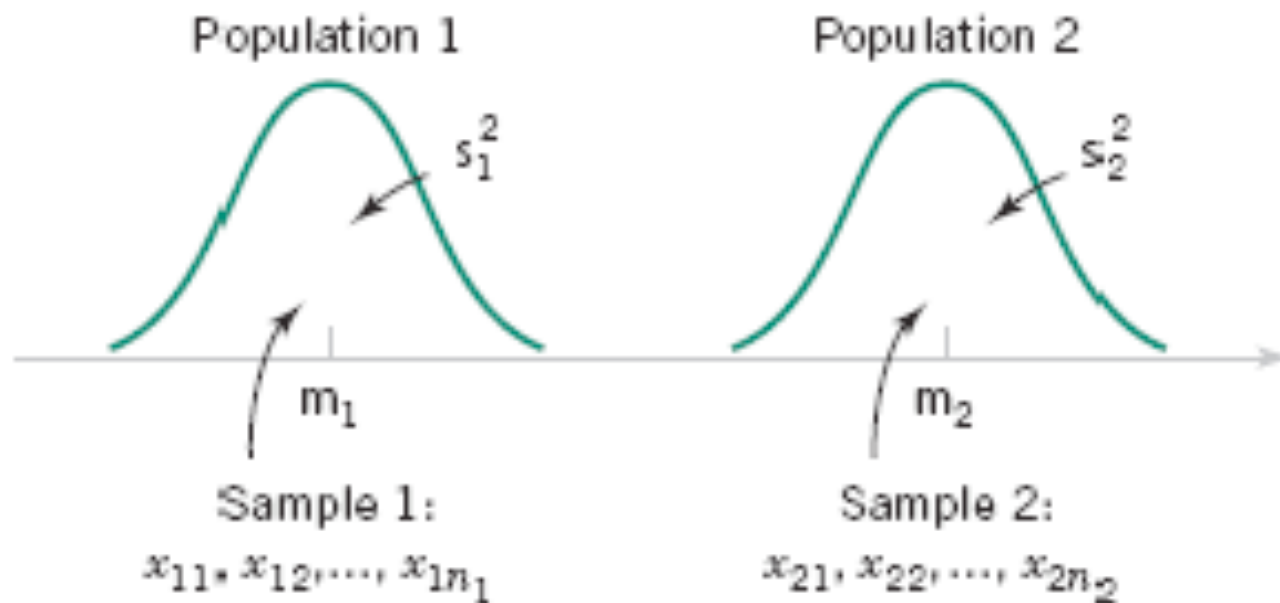
Figure 10-1 Two independent populations.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Assumptions

1. $X_{11}, X_{12}, \ldots, X_{1n_1}$ is a random sample from population 1.
2. $X_{21}, X_{22}, \ldots, X_{2n_2}$ is a random sample from population 2.
3. The two populations represented by X_1 and X_2 are independent.
4. Both populations are normal.

$$E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu_1 - \mu_2$$

$$V(\overline{X}_1 - \overline{X}_2) = V(\overline{X}_1) + V(\overline{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

The quantity

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \qquad (10\text{-}1)$$

has a $N(0, 1)$ distribution.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

10-2.1 Hypothesis Tests for a Difference in Means, Variances Known

Null hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$

Test statistic: $Z_0 = \dfrac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$ (10-2)

Alternative Hypotheses	Rejection Criterion
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	$z_0 < -z_\alpha$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-1

A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using $\alpha = 0.05$?

We apply the eight-step procedure to this problem as follows:

1. The quantity of interest is the difference in mean drying times, $\mu_1 - \mu_2$, and $\Delta_0 = 0$.
2. $H_0: \mu_1 - \mu_2 = 0$, or $H_0: \mu_1 = \mu_2$.
3. $H_1: \mu_1 > \mu_2$. We want to reject H_0 if the new ingredient reduces mean drying time.
4. $\alpha = 0.05$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-1

5. The test statistic is

$$z_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

where $\sigma_1^2 = \sigma_2^2 = (8)^2 = 64$ and $n_1 = n_2 = 10$.

6. Reject H_0: $\mu_1 = \mu_2$ if $z_0 > 1.645 = z_{0.05}$.

7. Computations: Since $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, the test statistic is

$$z_0 = \frac{121 - 112}{\sqrt{\dfrac{(8)^2}{10} + \dfrac{(8)^2}{10}}} = 2.52$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-1

8. Conclusion: Since $z_0 = 2.52 > 1.645$, we reject H_0: $\mu_1 = \mu_2$ at the $\alpha = 0.05$ level and conclude that adding the new ingredient to the paint significantly reduces the drying time. Alternatively, we can find the P-value for this test as

$$P\text{-value} = 1 - \Phi(2.52) = 0.0059$$

Therefore, H_0: $\mu_1 = \mu_2$ would be rejected at any significance level $\alpha \geq 0.0059$.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

10-2.2 Type II Error and Choice of Sample Size

- **Use of Operating Characteristic Curves**
- **Chart VII(a)-(d)**
- **Identical to 9-2.2 except**

$$d = \frac{|\mu_1 - \mu_2 - \Delta_0|}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{|\Delta - \Delta_0|}{\sqrt{\sigma_1^2 + \sigma_2^2}}$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

10-2.3 Confidence Interval on a Difference in Means, Variances Known

Definition

If $\bar{x}_1$ and $\bar{x}_2$ are the means of independent random samples of sizes n_1 and n_2 from two independent normal populations with known variances σ_1^2 and σ_2^2, respectively, a $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10\text{-}7)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-4

Tensile strength tests were performed on two different grades of aluminum spars used in manufacturing the wing of a commercial transport aircraft. From past experience with the spar manufacturing process and the testing procedure, the standard deviations of tensile strengths are assumed to be known. The data obtained are as follows: $n_1 = 10$, $\bar{x}_1 = 87.6$, $\sigma_1 = 1$, $n_2 = 12$, $\bar{x}_2 = 74.5$, and $\sigma_2 = 1.5$. If μ_1 and μ_2 denote the true mean tensile strengths for the two grades of spars, we may find a 90% confidence interval on the difference in mean strength $\mu_1 - \mu_2$ as follows:

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$87.6 - 74.5 - 1.645\sqrt{\frac{(1)^2}{10} + \frac{(1.5)^3}{12}} \leq \mu_1 - \mu_2 \leq 87.6 - 74.5 + 1.645\sqrt{\frac{(1^2)}{10} + \frac{(1.5)^2}{12}}$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-4

Therefore, the 90% confidence interval on the difference in mean tensile strength (in kilograms per square millimeter) is

$$12.22 \leq \mu_1 - \mu_2 \leq 13.98 \text{ (in kilograms per square millimeter)}$$

Notice that the confidence interval does not include zero, implying that the mean strength of aluminum grade 1 (μ_1) exceeds the mean strength of aluminum grade 2 (μ_2). In fact, we can state that we are 90% confident that the mean tensile strength of aluminum grade 1 exceeds that of aluminum grade 2 by between 12.22 and 13.98 kilograms per square millimeter.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Known

One-Sided Confidence Bounds

Upper Confidence Bound

$$\mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad (10\text{-}9)$$

Lower Confidence Bound

$$\bar{x}_1 - \bar{x}_2 - z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \qquad (10\text{-}10)$$

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-3.1 Hypotheses Tests for a Difference in Means, Variances Unknown

We wish to test:

$$H_0: \mu_1 - \mu_2 = \Delta_0$$
$$H_1: \mu_1 - \mu_2 \neq \Delta_0$$

Case 1: $\quad \sigma_1^2 = \sigma_2^2 = \sigma^2$

Case 2: $\quad \sigma_1^2 \neq \sigma_2^2$

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

The pooled estimator of σ^2:

The **pooled estimator of σ^2**, denoted by S_p^2, is defined by

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} \tag{10-12}$$

The pooled estimator is an **unbiased** estimator of σ^2

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Given the assumptions of this section, the quantity

$$T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

(10-13)

has a t distribution with $n_1 + n_2 - 2$ degrees of freedom.

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Definition: The Two-Sample or Pooled t-Test[*]

Null hypothesis:
$$H_0: \mu_1 - \mu_2 = \Delta_0$$

Test statistic:
$$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

(10-14)

Alternative Hypothesis	Rejection Criterion
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	$t_0 > t_{\alpha/2, n_1 + n_2 - 2}$ or $t_0 < -t_{\alpha/2, n_1 + n_2 - 2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	$t_0 > t_{\alpha, n_1 + n_2 - 2}$
$H_1: \mu_1 - \mu_2 < \Delta_0$	$t_0 < -t_{\alpha, n_1 + n_2 - 2}$

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently in use, but catalyst 2 is acceptable. Since catalyst 2 is cheaper, it should be adopted, providing it does not change the process yield. A test is run in the pilot plant and results in the data shown in Table 10-1. Is there any difference between the mean yields? Use $\alpha = 0.05$, and assume equal variances.

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Table 10-1   Catalyst Yield Data, Example 10-5

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown
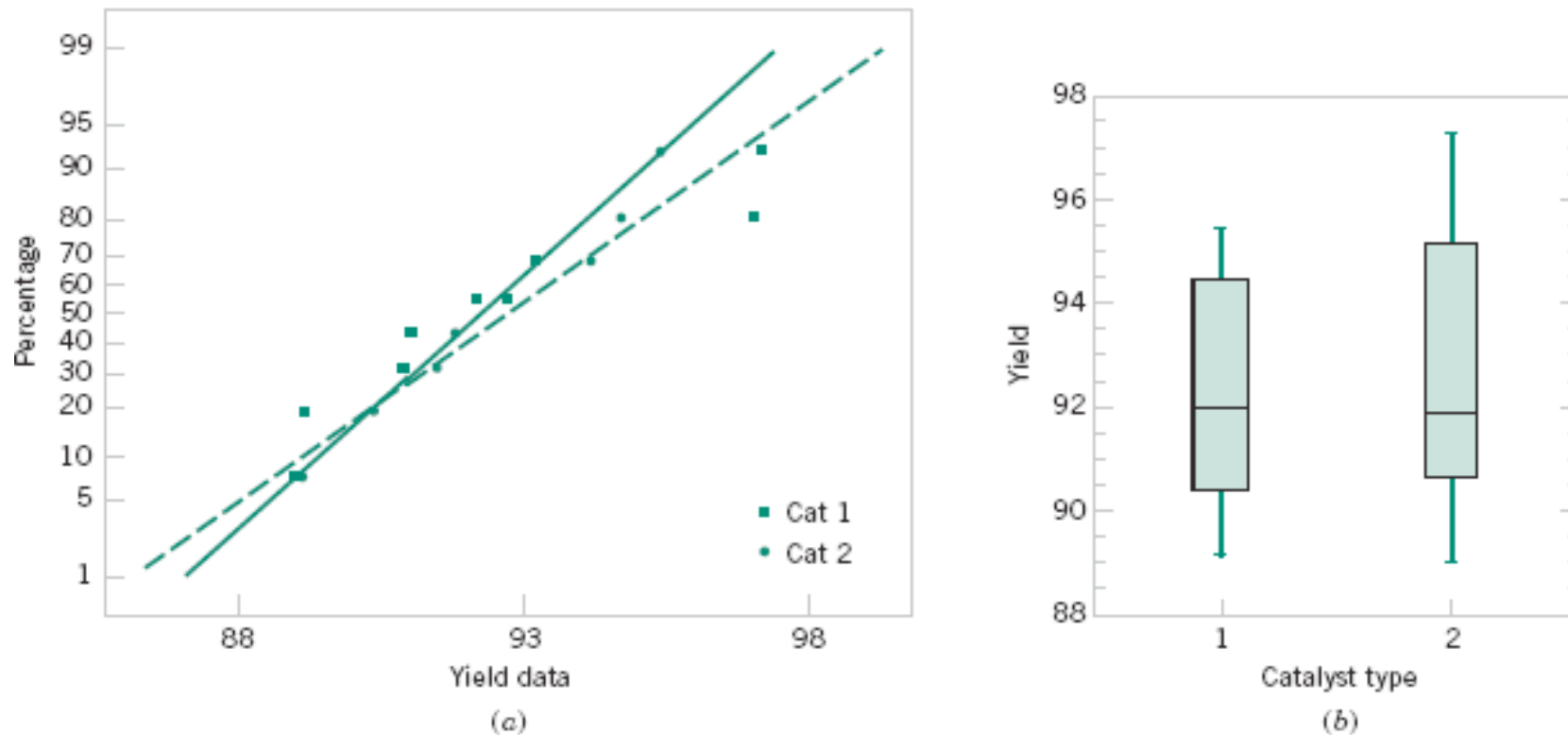
Figure 10-2 Normal probability plot and comparative box plot for the catalyst yield data in Example 10-5. (a) Normal probability plot, (b) Box plots.

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

The solution using the eight-step hypothesis-testing procedure is as follows:

1. The parameters of interest are μ_1 and μ_2, the mean process yield using catalysts 1 and 2, respectively, and we want to know if $\mu_1 - \mu_2 = 0$.

2. $H_0: \mu_1 - \mu_2 = 0$, or $H_0: \mu_1 = \mu_2$

3. $H_1: \mu_1 \neq \mu_2$

4. $\alpha = 0.05$

5. The test statistic is

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{s_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

6. Reject H_0 if $t_0 > t_{0.025,14} = 2.145$ or if $t_0 < -t_{0.025,14} = -2.145$.

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

7. Computations: From Table 10-1 we have $\bar{x}_1 = 92.255$, $s_1 = 2.39$, $n_1 = 8$, $\bar{x}_2 = 92.733$, $s_2 = 2.98$, and $n_2 = 8$. Therefore

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(7)(2.39)^2 + 7(2.98)^2}{8 + 8 - 2} = 7.30$$

$$s_p = \sqrt{7.30} = 2.70$$

and

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{2.70\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{92.255 - 92.733}{2.70\sqrt{\dfrac{1}{8} + \dfrac{1}{8}}} = -0.35$$

8. Conclusions: Since $-2.145 < t_0 = -0.35 < 2.145$, the null hypothesis cannot be rejected. That is, at the 0.05 level of significance, we do not have strong evidence to conclude that catalyst 2 results in a mean yield that differs from the mean yield when catalyst 1 is used.

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Case 2: $\sigma_1^2 \neq \sigma_2^2$

$$T_0^* = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} \qquad (10\text{-}15)$$

is distributed approximately as t with degrees of freedom given by

$$v = \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{(S_1^2/n_1)^2}{n_1 - 1} + \dfrac{(S_2^2/n_2)^2}{n_2 - 1}} \qquad (10\text{-}16)$$

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6

Arsenic concentration in public drinking water supplies is a potential health risk. An article in the *Arizona Republic* (Sunday, May 27, 2001) reported drinking water arsenic concentrations in parts per billion (ppb) for 10 methropolitan Phoenix communities and 10 communities in rural Arizona. The data follow:

Metro Phoenix ($\bar{x}_1 = 12.5, s_1 = 7.63$)	Rural Arizona ($\bar{x}_2 = 27.5, s_2 = 15.3$)
Phoenix, 3	Rimrock, 48
Chandler, 7	Goodyear, 44
Gilbert, 25	New River, 40
Glendale, 10	Apachie Junction, 38
Mesa, 15	Buckeye, 33
Paradise Valley, 6	Nogales, 21
Peoria, 12	Black Canyon City, 20
Scottsdale, 25	Sedona, 12
Tempe, 15	Payson, 1
Sun City, 7	Casa Grande, 18

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

We wish to determine it there is any difference in mean arsenic concentrations between metropolitan Phoenix communities and communities in rural Arizona. Figure 10-3 shows a normal probability plot for the two samples of arsenic concentration. The assumption of normality appears quite reasonable, but since the slopes of the two straight lines are very different, it is unlikely that the population variances are the same.

Applying the eight-step procedure gives the following:

1. The parameters of interest are the mean arsenic concentrations for the two geographic regions, say, μ_1 and μ_2, and we are interested in determining whether $\mu_1 - \mu_2 = 0$.
2. $H_0: \mu_1 - \mu_2 = 0$, or $H_0: \mu_1 = \mu_2$
3. $H_1: \mu_1 \neq \mu_2$
4. $\alpha = 0.05$ (say)

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

Figure 10-3 Normal probability plot of the arsenic concentration data from Example 10-6.
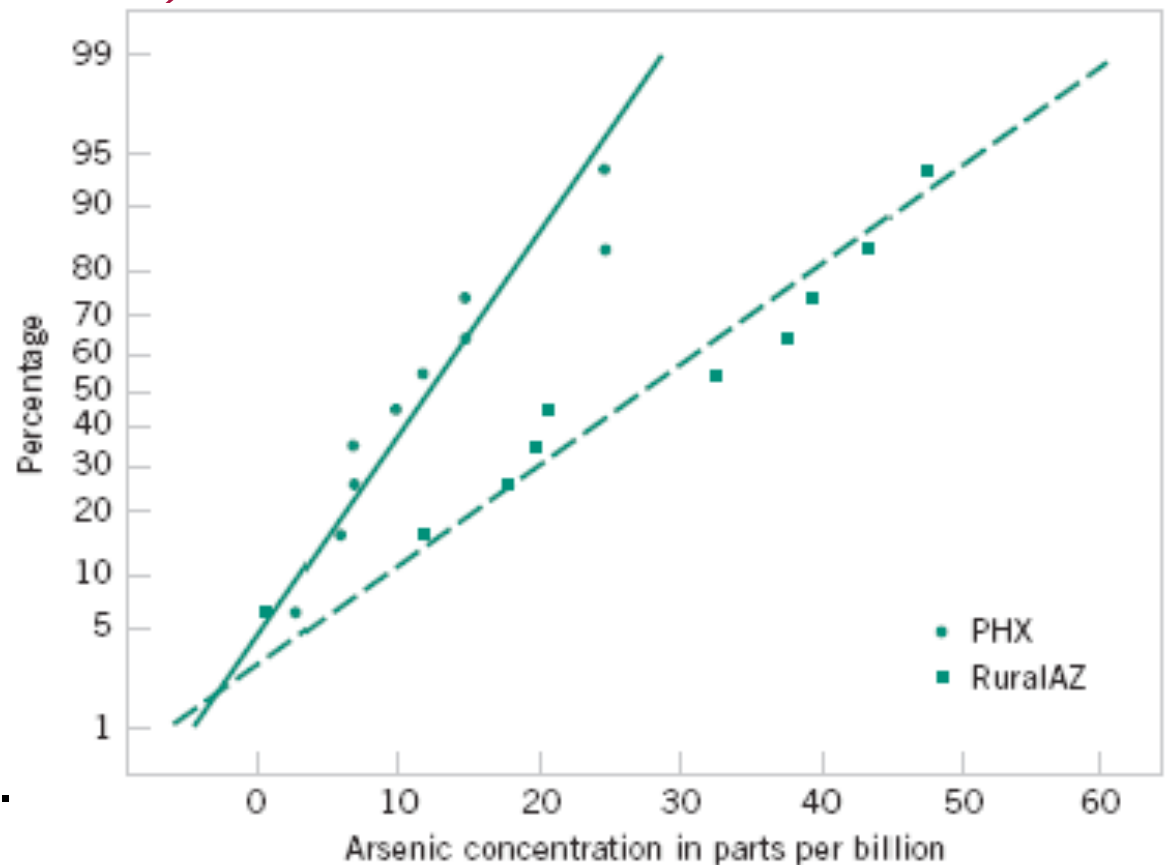
10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

5. The test statistic is

$$t_0^* = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

6. The degrees of freedom on t_0^* are found from Equation 10-16 as

$$v = \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}} = \frac{\left[\dfrac{(7.63)^2}{10} + \dfrac{(15.3)^2}{10}\right]^2}{\dfrac{[(7.63)^2/10]^2}{9} + \dfrac{[(15.3)^2/10]^2}{9}} = 13.2 \approx 13$$

Therefore, using $\alpha = 0.05$, we would reject H_0: $\mu_1 = \mu_2$ if $t_0^* > t_{0.025, 13} = 2.160$ or if $t_0^* < -t_{0.025, 13} = -2.160$

10-3 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

7. Computations: Using the sample data we find

$$t_0^* = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{12.5 - 27.5}{\sqrt{\frac{(7.63)^2}{10} + \frac{(15.3)^2}{10}}} = -2.77$$

8. Conclusions: Because $t_0^* = -2.77 < t_{0.025,13} = -2.160$, we reject the null hypothesis. Therefore, there is evidence to conclude that mean arsenic concentration in the drinking water in rural Arizona is different from the mean arsenic concentration in metropolitan Phoenix drinking water. Furthermore, the mean arsenic concentration is higher in rural Arizona communities. The P-value for this test is approximately $P = 0.016$.

10-3.3 Confidence Interval on the Difference in Means, Variance Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

If $\bar{x}_1, \bar{x}_2$, s_1^2 and s_2^2 are the sample means and variances of two random samples of sizes n_1 and n_2, respectively, from two independent normal populations with unknown but equal variances, then a $100(1 - \alpha)\%$ confidence interval on the difference in means $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,n_1+n_2-2} \, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$\leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,n_1+n_2-2} \, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \qquad (10\text{-}19)$$

where $s_p = \sqrt{[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/(n_1 + n_2 - 2)}$ is the pooled estimate of the common population standard deviation, and $t_{\alpha/2,n_1+n_2-2}$ is the upper $\alpha/2$ percentage point of the t distribution with $n_1 + n_2 - 2$ degrees of freedom.

10-3.3 Confidence Interval on the Difference in Means, Variance Unknown

Case 2: $\sigma_1^2 \neq \sigma_2^2$

If $\bar{x}_1, \bar{x}_2, s_1^2,$ and s_2^2 are the means and variances of two random samples of sizes n_1 and n_2, respectively, from two independent normal populations with unknown and unequal variances, an approximate $100(1 - \alpha)\%$ confidence interval on the difference in means $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \quad (10\text{-}20)$$

where v is given by Equation 10-16 and $t_{\alpha/2,v}$ is the upper $\alpha/2$ percentage point of the t distribution with v degrees of freedom.

10-4 Paired *t*-Test

• A special case of the two-sample *t*-tests of Section 10-3 occurs when the observations on the two populations of interest are collected in **pairs**.

• Each pair of observations, say (X_{1j} , X_{2j}), is taken under homogeneous conditions, but these conditions may change from one pair to another.

• The test procedure consists of analyzing the **differences** between two observations from each pair.

10-4 Paired t-Test

The Paired t-Test

Null hypothesis: H_0: $\mu_D = \Delta_0$

Test statistic: $T_0 = \dfrac{\overline{D} - \Delta_0}{S_D/\sqrt{n}}$ (10-22)

Alternative Hypothesis	Rejection Region
H_1: $\mu_D \neq \Delta_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
H_1: $\mu_D > \Delta_0$	$t_0 > t_{\alpha,n-1}$
H_1: $\mu_D < \Delta_0$	$t_0 < -t_{\alpha,n-1}$

In Equation 10-22, $\overline{D}$ is the sample average of the n differences $D_1, D_2, ..., D_n$, and S_D is the sample standard deviation of these differences.

10-4 Paired *t*-Test

Example 10-9

An article in the *Journal of Strain Analysis* (1983, Vol. 18, No. 2) compares several methods for predicting the shear strength for steel plate girders. Data for two of these methods, the Karlsruhe and Lehigh procedures, when applied to nine specific girders, are shown in Table 10-2. We wish to determine whether there is any difference (on the average) between the two methods.

Table 10-2 Strength Predictions for Nine Steel Plate Girders (Predicted Load/Observed Load)

Girder	Karlsruhe Method	Lehigh Method	Difference d_j
S1/1	1.186	1.061	0.119
S2/1	1.151	0.992	0.159
S3/1	1.322	1.063	0.259
S4/1	1.339	1.062	0.277
S5/1	1.200	1.065	0.138
S2/1	1.402	1.178	0.224
S2/2	1.365	1.037	0.328
S2/3	1.537	1.086	0.451
S2/4	1.559	1.052	0.507

10-4 Paired *t*-Test

Example 10-9

The eight-step procedure is applied as follows:

1. The parameter of interest is the difference in mean shear strength between the two methods, say, $\mu_D = \mu_1 - \mu_2 = 0$.

2. H_0: $\mu_D = 0$

3. H_1: $\mu_D \neq 0$

4. $\alpha = 0.05$

5. The test statistic is

$$t_0 = \frac{\bar{d}}{s_D/\sqrt{n}}$$

6. Reject H_0 if $t_0 > t_{0.025,8} = 2.306$ or if $t_0 < -t_{0.025,8} = -2.306$.

10-4 Paired t-Test

Example 10-9

7. Computations: The sample average and standard deviation of the differences d_j are $\bar{d} = 0.2736$ and $s_D = 0.1356$, so the test statistic is

$$t_0 = \frac{\bar{d}}{s_D/\sqrt{n}} = \frac{0.2736}{0.1356/\sqrt{9}} = 6.05$$

8. Conclusions: Since $t_0 = 6.05 > 2.306$, we conclude that the strength prediction methods yield different results. Specifically, the data indicate that the Karlsruhe method produces, on the average, higher strength predictions than does the Lehigh method. The P-value for $t_0 = 6.05$ is $P = 0.0002$, so the test statistic is well into the critical region.

10-4 Paired *t*-Test

Paired Versus Unpaired Comparisons

$$T_0 = \frac{\bar{D} - \Delta_0}{S_D / \sqrt{n}}$$

$$T_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{S_p \sqrt{\frac{1}{n} + \frac{1}{n}}}$$

So how do we decide to conduct the experiment? Should we pair the observations or not? Although there is no general answer to this question, we can give some guidelines based on the above discussion.

1. If the experimental units are relatively homogeneous (small σ) and the correlation within pairs is small, the gain in precision attributable to pairing will be offset by the loss of degrees of freedom, so an independent-sample experiment should be used.

2. If the experimental units are relatively heterogeneous (large σ) and there is large positive correlation within pairs, the paired experiment should be used. Typically, this case occurs when the experimental units are the *same* for both treatments; as in Example 10-9, the same girders were used to test the two methods.

10-4 Paired *t*-Test

A Confidence Interval for μ_D

If $\bar{d}$ and s_D are the sample mean and standard deviation of the difference of n random pairs of normally distributed measurements, a **$100(1 - \alpha)\%$ confidence interval on the difference in means $\mu_D = \mu_1 - \mu_2$** is

$$\bar{d} - t_{\alpha/2,n-1}s_D/\sqrt{n} \le \mu_D \le \bar{d} + t_{\alpha/2,n-1}s_D/\sqrt{n} \qquad (10\text{-}23)$$

where $t_{\alpha/2,n-1}$ is the upper $\alpha/2\%$ point of the t-distribution with $n - 1$ degrees of freedom.