14 Design of Experiments with Several Factors

CHAPTER OUTLINE

14-1 Introduction

- An **experiment** is a test or series of tests.
- The **design** of an experiment plays a major role in the eventual solution of the problem.
- In a **factorial experimental design**, experimental trials (or runs) are performed at all combinations of the factor levels.
- The **analysis of variance** (ANOVA) will be used as one of the primary tools for statistical data analysis.

14-2 Factorial Experiments

Definition

By a **factorial experiment** we mean that in each complete trial or replicate of the experiment all possible combinations of the levels of the factors are investigated.

Table 14-1 A Factorial Experiment with Two Factors

	Factor B	
Factor A	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	40

Table 14-2 A Factorial Experiment with Interaction

	Factor B	
Factor A	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	0

14-2 Factorial Experiments

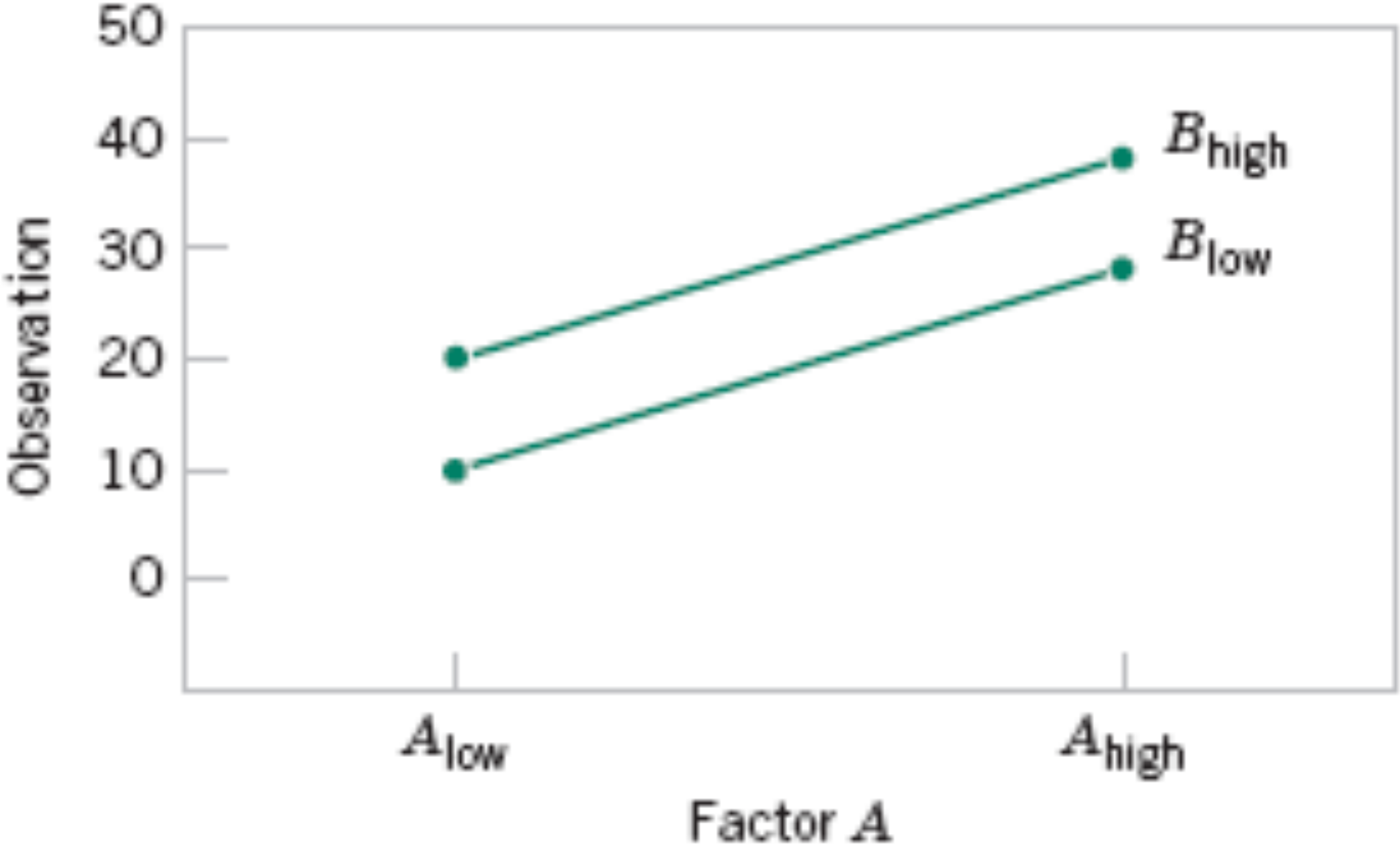

Figure 14-3 Factorial Experiment, no interaction.

14-2 Factorial Experiments

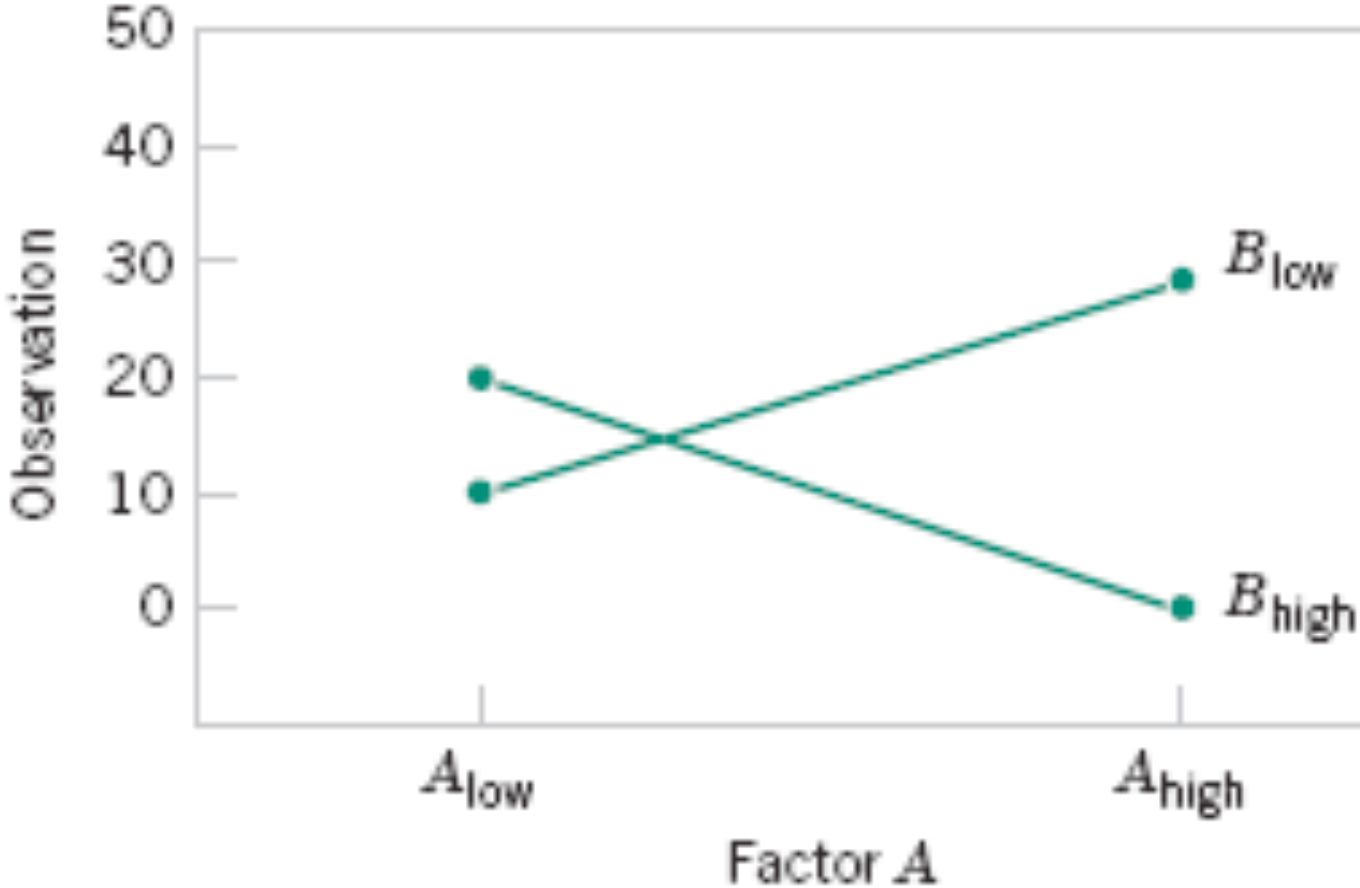

Figure 14-4 Factorial Experiment, with interaction.

14-2 Factorial Experiments

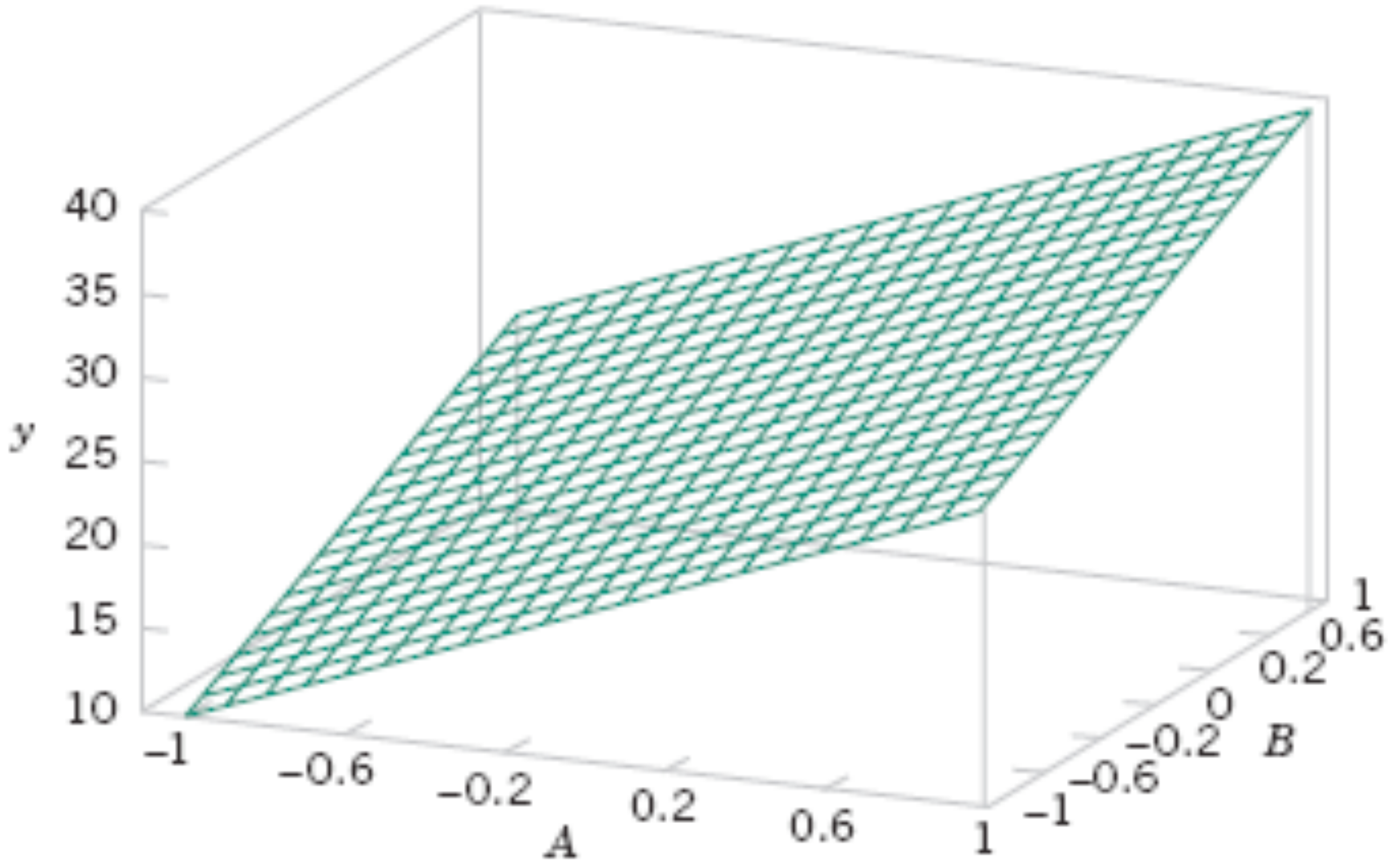

Figure 14-5 Three-dimensional surface plot of the data from Table 14-1, showing main effects of the two factors *A* and *B*.

14-2 Factorial Experiments

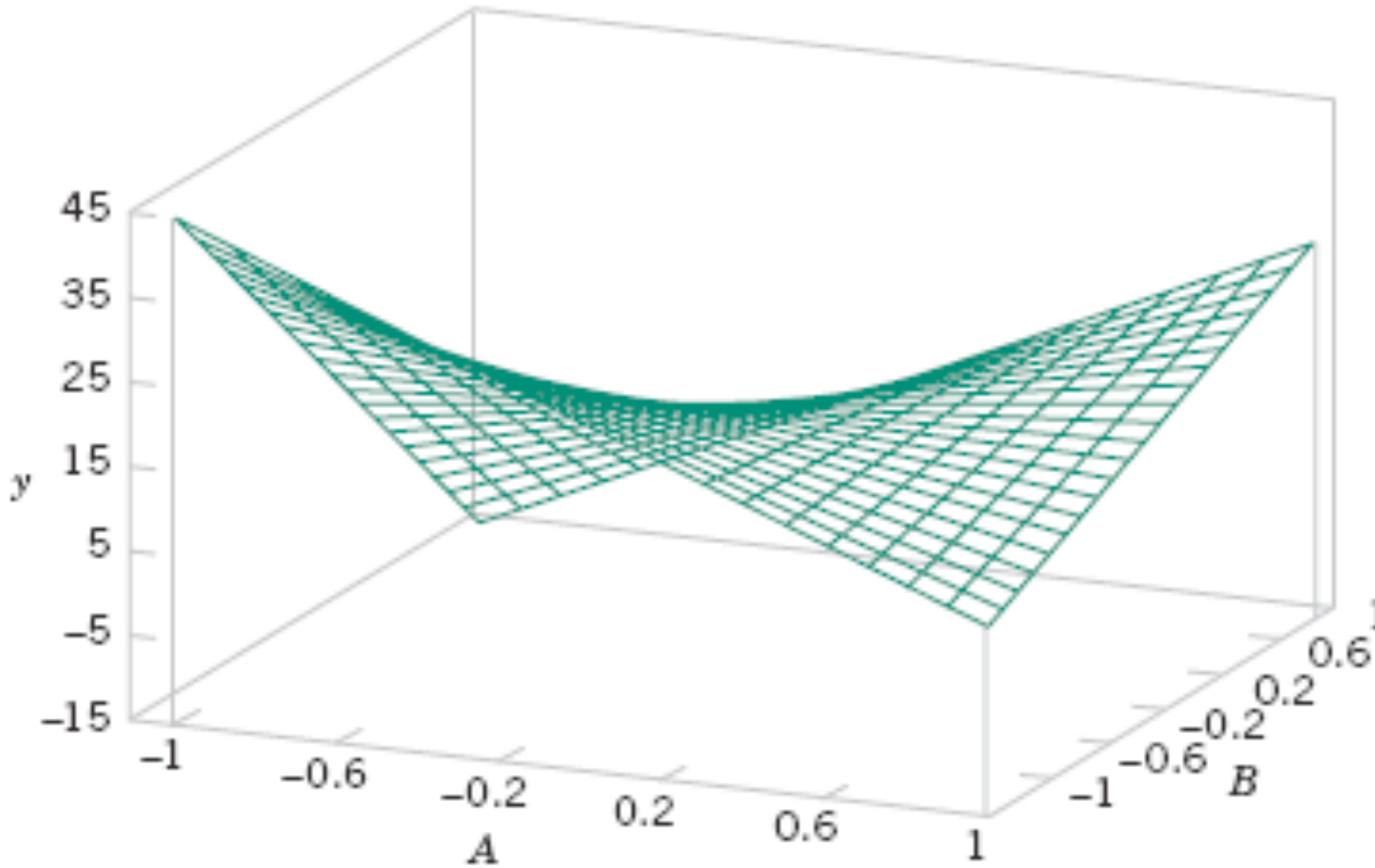

Figure 14-6 Three-dimensional surface plot of the data from Table 14-2, showing main effects of the *A* and *B* interaction.

14-2 Factorial Experiments

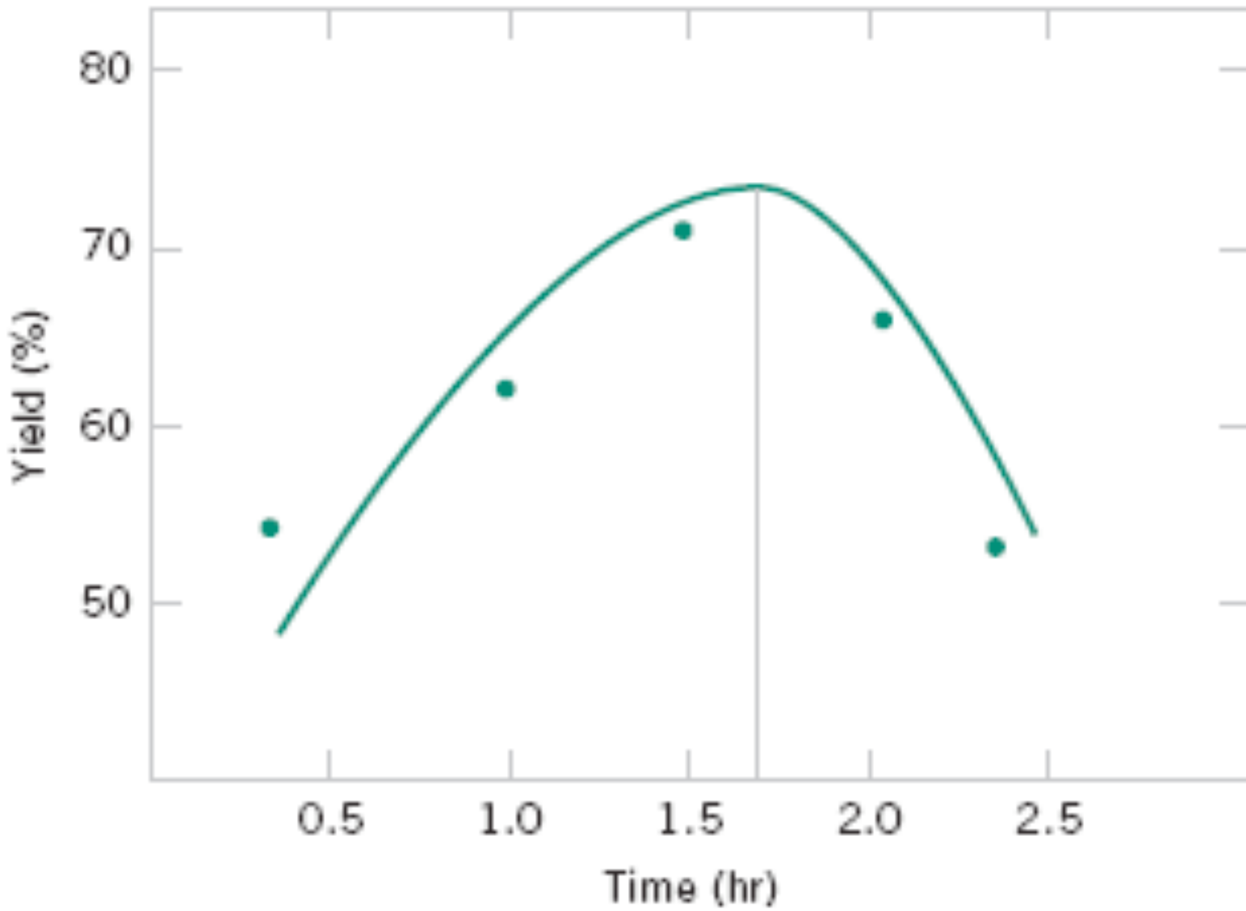

Figure 14-7 Yield versus reaction time with temperature constant at 155º F.

14-2 Factorial Experiments

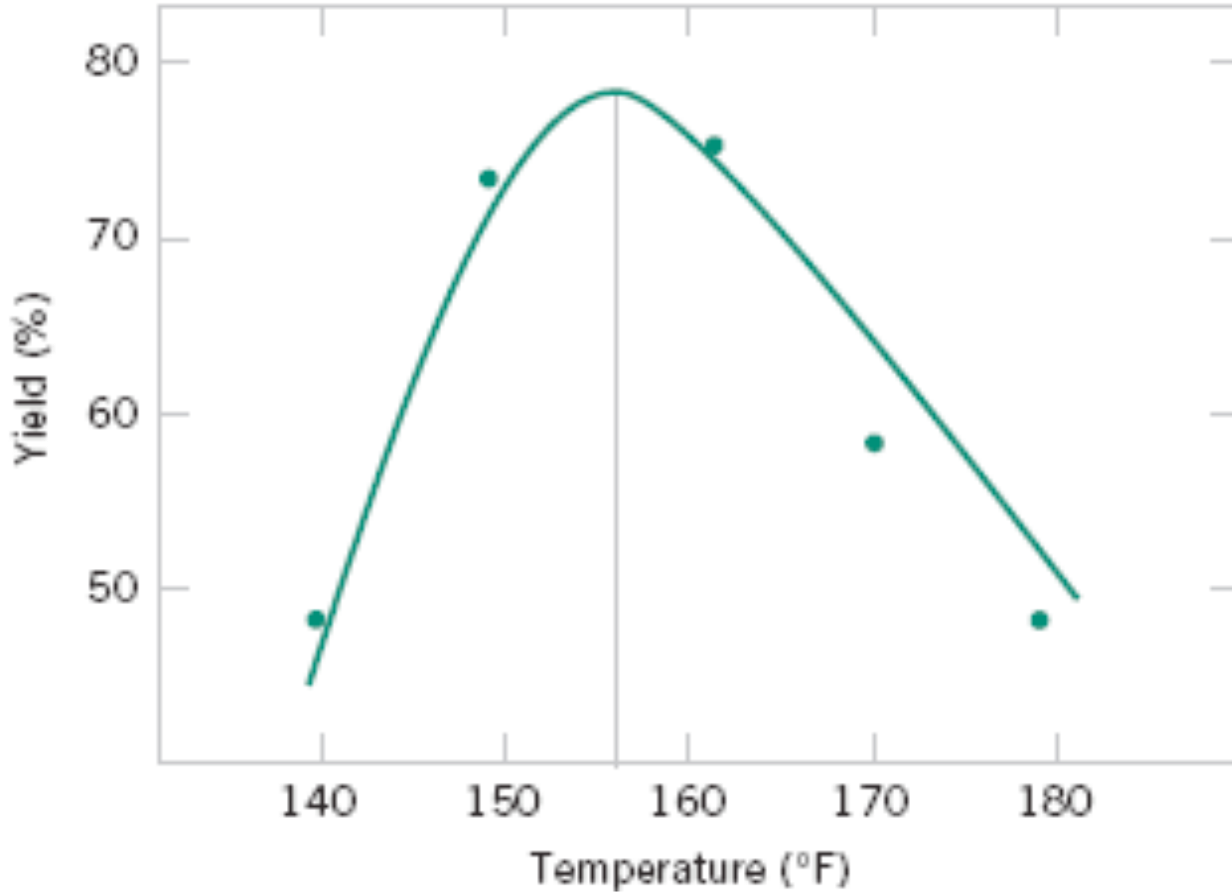

Figure 14-8 Yield versus temperature with reaction time constant at 1.7 hours.

14-2 Factorial Experiments

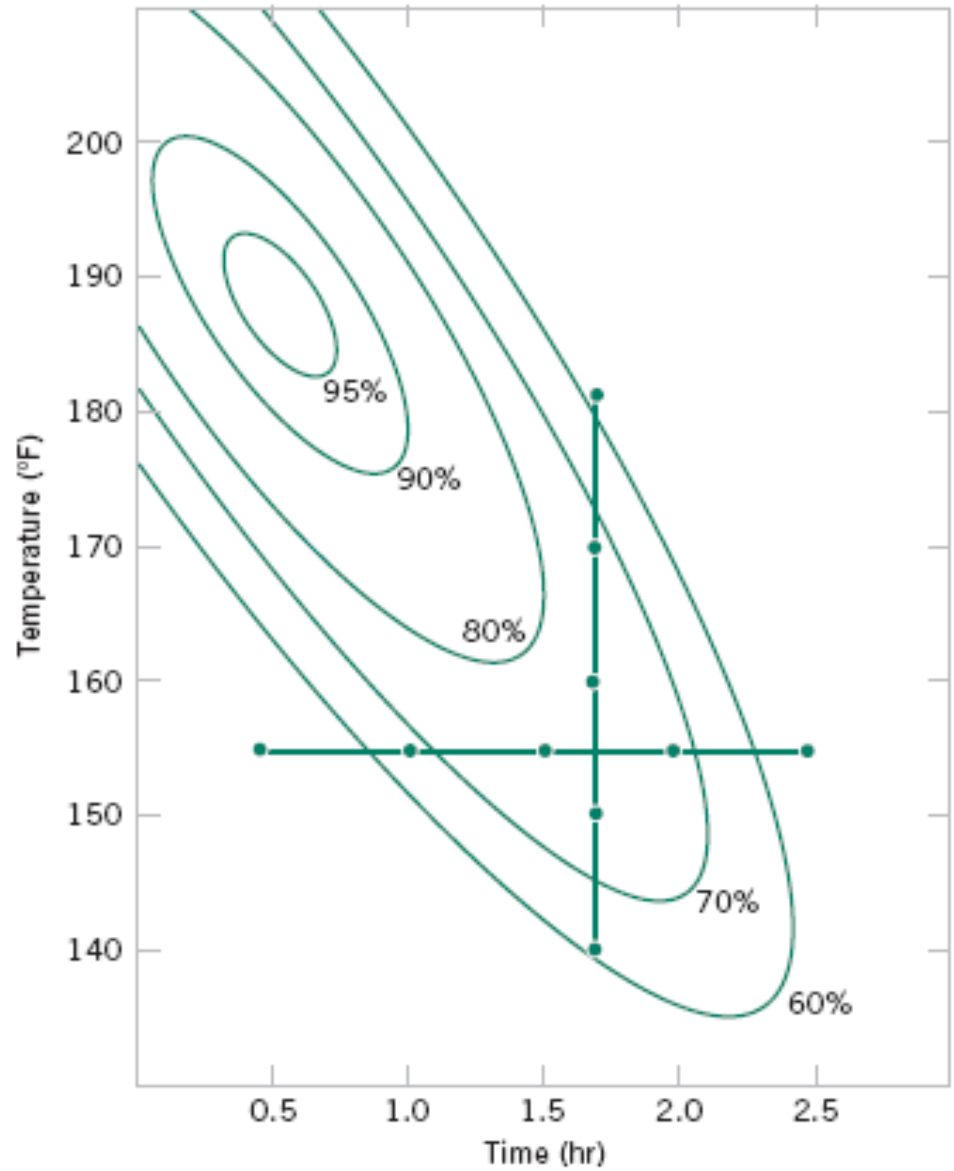

Figure 14-9 Optimization experiment using the one-factor-at-a-time method.

14-3 Two-Factor Factorial Experiments

Table 14-3 Data Arrangement for a Two-Factor Factorial Design

		Factor B				Totals	Averages
		1	2	$\cdots$	b		
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$	$y_{1\cdot\cdot}$	$\bar{y}_{1\cdot\cdot}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$	$y_{2\cdot\cdot}$	$\bar{y}_{2\cdot\cdot}$
	$\vdots$						
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$	$y_{a\cdot\cdot}$	$\bar{y}_{a\cdot\cdot}$
Totals		$y_{\cdot 1\cdot}$	$y_{\cdot 2\cdot}$		$y_{\cdot b\cdot}$	$y_{\cdots}$	
Averages		$\bar{y}_{\cdot 1\cdot}$	$\bar{y}_{\cdot 2\cdot}$		$\bar{y}_{\cdot b\cdot}$		$\bar{y}_{\cdots}$

14-3 Two-Factor Factorial Experiments

The observations may be described by the linear statistical model:

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

$$y_{i\cdot\cdot} = \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{i\cdot\cdot} = \frac{y_{i\cdot\cdot}}{bn} \qquad i = 1, 2, \ldots, a$$

$$y_{\cdot j\cdot} = \sum_{i=1}^{a} \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{\cdot j\cdot} = \frac{y_{\cdot j\cdot}}{an} \qquad j = 1, 2, \ldots, b$$

$$y_{ij\cdot} = \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{ij\cdot} = \frac{y_{ij\cdot}}{n} \qquad \begin{array}{l} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{array}$$

$$y_{\cdots} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{\cdots} = \frac{y_{\cdots}}{abn}$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

The hypotheses that we will test are as follows:

1. $H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$ (no main effect of factor A)
   H_1: at least one $\tau_i \neq 0$
2. $H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$ (no main effect of factor B)
   H_1: at least one $\beta_j \neq 0$
3. $H_0: (\tau\beta)_{11} = (\tau\beta)_{12} = \cdots = (\tau\beta)_{ab} = 0$ (no interaction)
   H_1: at least one $(\tau\beta)_{ij} \neq 0$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

The sum of squares identity for a two-factor ANOVA is

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2$$
$$+ an\sum_{j=1}^{b}(\bar{y}_{.j.}-\bar{y}_{...})^2$$
$$+ n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$$
$$+ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2 \qquad (14\text{-}3)$$

or symbolically,

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E \qquad (14\text{-}4)$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

To test H_0: $\tau_i = 0$ use the ratio

$$F_0 = \frac{MS_A}{MS_E}$$

To test H_0: $\beta_j = 0$ use the ratio

$$F_0 = \frac{MS_B}{MS_E}$$

To test H_0: $(\tau\beta)_{ij} = 0$ use the ratio

$$F_0 = \frac{MS_{AB}}{MS_E}$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Definition

Computing formulas for the sums of squares in a two-factor analysis of variance.

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\cdots}^2}{abn} \qquad (14\text{-}5)$$

$$SS_A = \sum_{i=1}^{a} \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdots}^2}{abn} \qquad (14\text{-}6)$$

$$SS_B = \sum_{j=1}^{b} \frac{y_{\cdot j\cdot}^2}{an} - \frac{y_{\cdots}^2}{abn} \qquad (14\text{-}7)$$

$$SS_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij\cdot}^2}{n} - \frac{y_{\cdots}^2}{abn} - SS_A - SS_B \qquad (14\text{-}8)$$

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B \qquad (14\text{-}9)$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Table 14-4 ANOVA Table for a Two-Factor Factorial, Fixed-Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a-1$	$MS_A = \dfrac{SS_A}{a-1}$	$\dfrac{MS_A}{MS_E}$
B treatments	SS_B	$b-1$	$MS_B = \dfrac{SS_B}{b-1}$	$\dfrac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$\dfrac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$		
Total	SS_T	$abn-1$	$MS_E = \dfrac{SS_E}{ab(n-1)}$	

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

Aircraft primer paints are applied to aluminum surfaces by two methods: dipping and spraying. The purpose of the primer is to improve paint adhesion, and some parts can be primed using either application method. The process engineering group responsible for this operation is interested in learning whether three different primers differ in their adhesion properties. A factorial experiment was performed to investigate the effect of paint primer type and application method on paint adhesion. For each combination of primer type and application method, three specimens were painted, then a finish paint was applied, and the adhesion force was measured. The data from the experiment are shown in Table 14-5. The circled numbers in the cells are the cell totals $y_{ij.}$. The sums of squares required to perform the ANOVA are computed as follows:

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

Table 14-5 Adhesion Force Data for Example 14-1

Primer Type	Dipping		Spraying		$y_{i\cdot\cdot}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{\cdot j\cdot}$	40.2		49.6		$89.8 = y_{\cdots}$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn}$$

$$= (4.0)^2 + (4.5)^2 + \cdots + (5.0)^2 - \frac{(89.8)^2}{18} = 10.72$$

$$SS_{\text{types}} = \sum_{i=1}^{a} \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}$$

$$= \frac{(28.7)^2 + (34.1)^2 + (27.0)^2}{6} - \frac{(89.8)^2}{18} = 4.58$$

$$SS_{\text{methods}} = \sum_{j=1}^{b} \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}$$

$$= \frac{(40.2)^2 + (49.6)^2}{9} - \frac{(89.8)^2}{18} = 4.91$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

$$SS_{\text{interaction}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{y_{ij\cdot}^2}{n} - \frac{y_{\cdots}^2}{abn} - SS_{\text{types}} - SS_{\text{methods}}$$

$$= \frac{(12.8)^2 + (15.9)^2 + (11.5)^2 + (15.9)^2 + (18.2)^2 + (15.5)^2}{3}$$

$$- \frac{(89.8)^2}{18} - 4.58 - 4.91 = 0.24$$

and

$$SS_E = SS_T - SS_{\text{types}} - SS_{\text{methods}} - SS_{\text{interaction}}$$

$$= 10.72 - 4.58 - 4.91 - 0.24 = 0.99$$

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

The ANOVA is summarized in Table 14-6. The experimenter has decided to use $\alpha = 0.05$. Since $f_{0.05,2,12} = 3.89$ and $f_{0.05,1,12} = 4.75$, we conclude that the main effects of primer type and application method affect adhesion force. Furthermore, since $1.5 < f_{0.05,2,12}$, there is no indication of interaction between these factors. The last column of Table 14-6 shows the P-value for each F-ratio. Notice that the P-values for the two test statistics for the main effects are considerably less than 0.05, while the P-value for the test statistic for the interaction is greater than 0.05.

A graph of the cell adhesion force averages $\{\bar{y}_{ij\cdot}\}$ versus levels of primer type for each application method is shown in Fig. 14-8. The no-interaction conclusion is obvious in this graph, because the two lines are nearly parallel. Furthermore, since a large response indicates greater adhesion force, we conclude that spraying is the best application method and that primer type 2 is most effective.

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

Table 14-6 ANOVA for Example 14-1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
Primer types	4.58	2	2.29	28.63	$2.7 \times$ E-5
Application methods	4.91	1	4.91	61.38	$4.7 \times$ E-7
Interaction	0.24	2	0.12	1.50	0.2621
Error	0.99	12	0.08		
Total	10.72	17			

14-3 Two-Factor Factorial Experiments

14-3.1 Statistical Analysis of the Fixed-Effects Model

Example 14-1

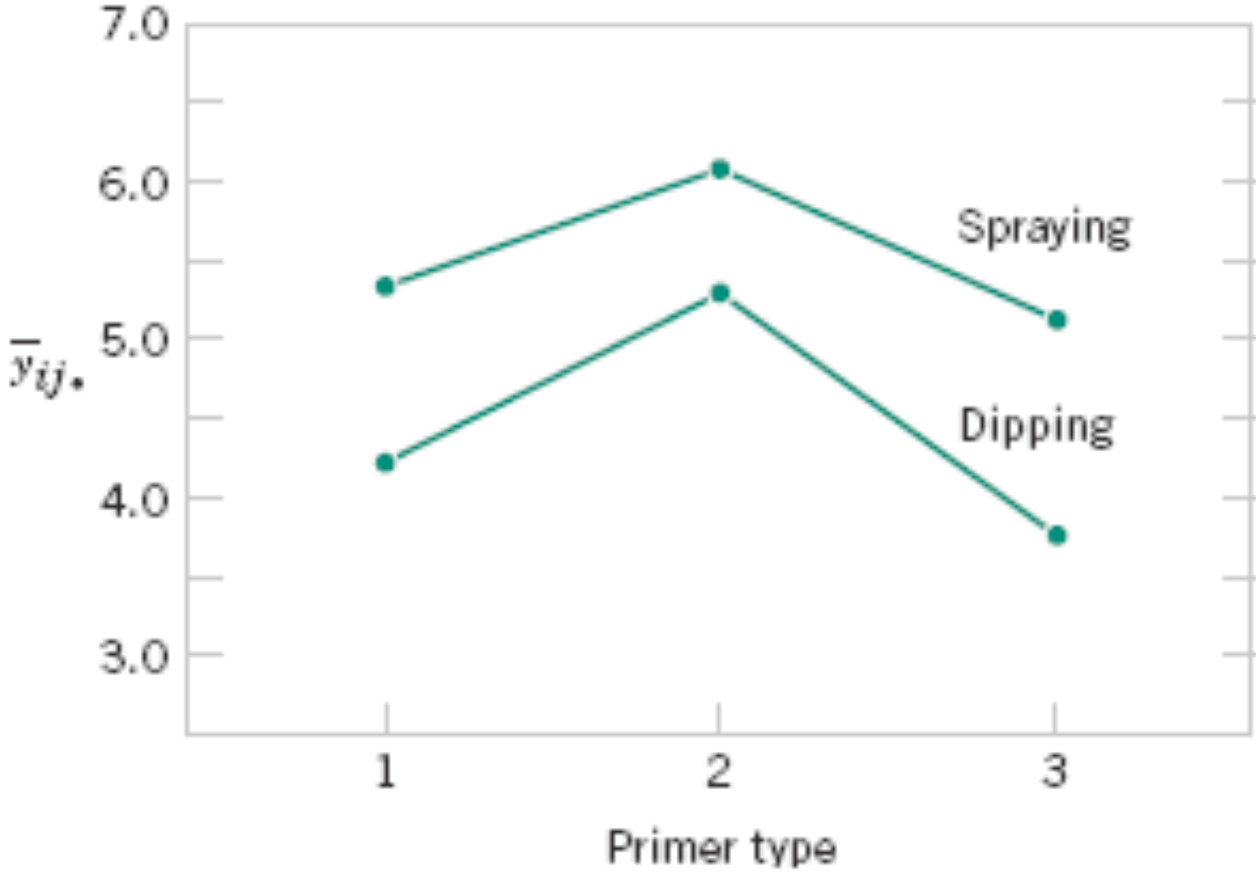

Figure 14-10 Graph of average adhesion force versus primer types for both application methods.

R commands and outputs

Example 14-1: enter data by row

```
> Adhesion=c(4.0, 4.5, 4.3, 5.4, 4.9, 5.6,  5.6, 4.9, 5.4, 5.8, 6.1, 6.3,  3.8,
   3.7, 4.0, 5.5, 5.0, 5.0)
> Primer=c(1,1,1,1,1,1, 2,2,2,2,2,2, 3,3,3,3,3,3)
> Method=c(1,1,1,2,2,2, 1,1,1,2,2,2, 1,1,1,2,2,2)  # 1=Dipping, 2=Spraying
> g=lm(Adhesion ~ as.factor(Primer) * as.factor(Method))
> anova(g)
Response: Adhesion
                                    Df Sum Sq Mean Sq F value    Pr(>F)
as.factor(Primer)                    2 4.5811  2.2906 27.8581 3.097e-05
as.factor(Method)                    1 4.9089  4.9089 59.7027 5.357e-06
as.factor(Primer):as.factor(Method)  2 0.2411  0.1206  1.4662    0.2693
Residuals                           12 0.9867  0.0822

> interaction.plot(Primer, Method, Adhesion)
```

See ch14.R for more commands

14-3 Two-Factor Factorial Experiments

14-3.2 Model Adequacy Checking

Table 14-8 Residuals for the Aircraft Primer Experiment in Example 14-1

	Application Method	
Primer Type	Dipping	Spraying
1	−0.27, 0.23, 0.03	0.10, −0.40, 0.30
2	0.30, −0.40, 0.10	−0.27, 0.03, 0.23
3	−0.03, −0.13, 0.17	0.33, −0.17, −0.17

14-3 Two-Factor Factorial Experiments

14-3.2 Model Adequacy Checking

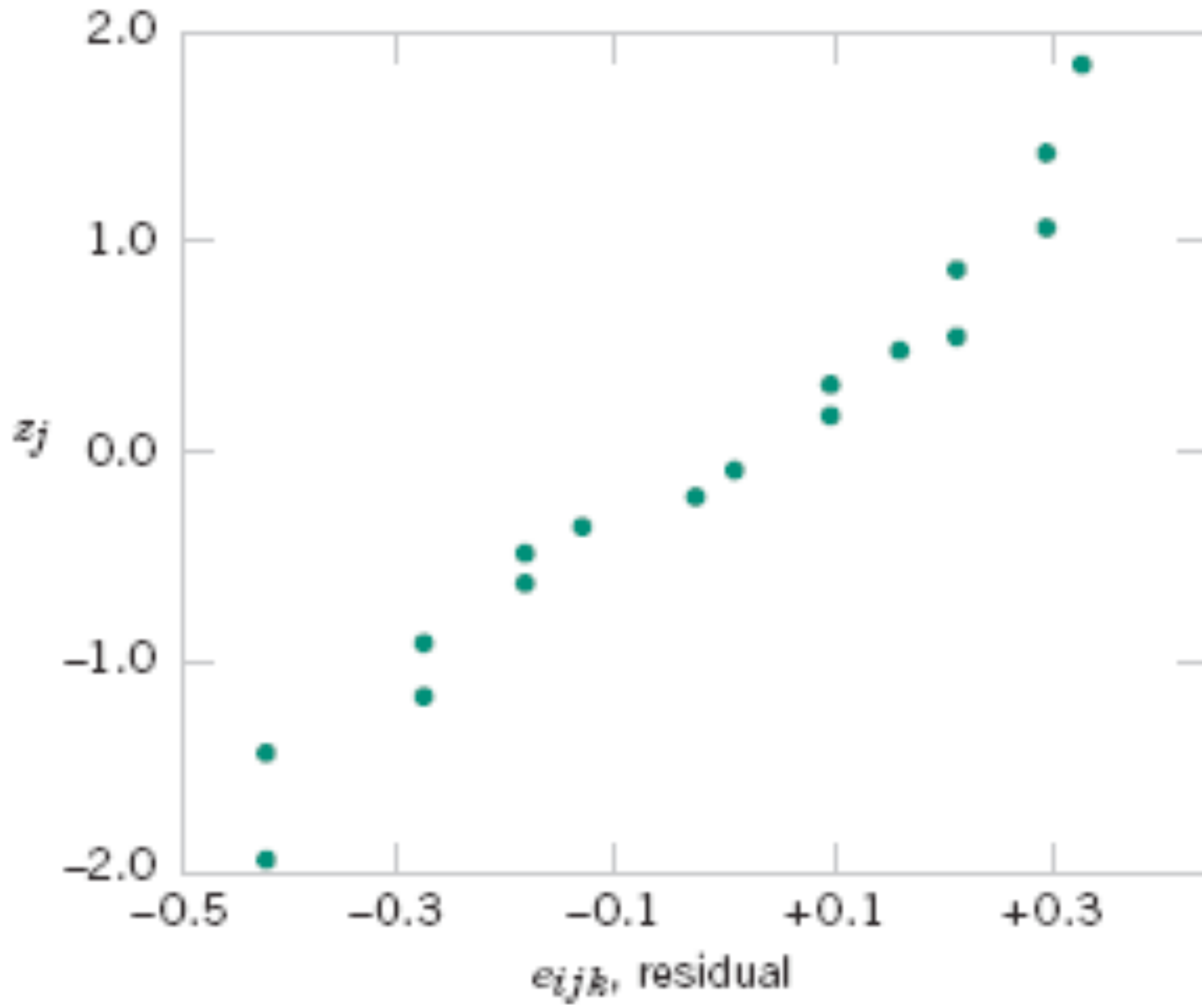

Figure 14-11
Normal probability plot of the residuals from Example 14-1

14-3 Two-Factor Factorial Experiments

14-3.2 Model Adequacy Checking

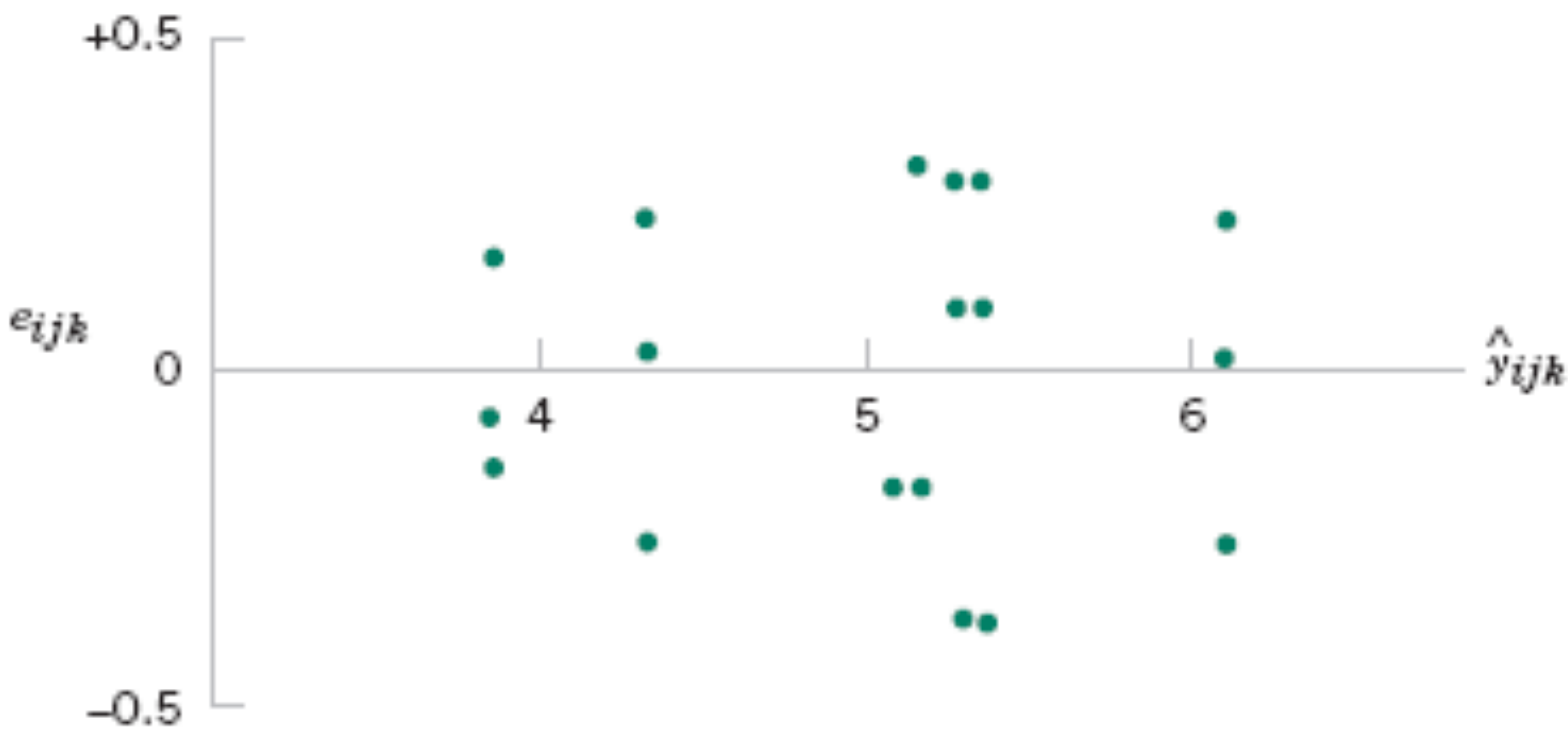

Figure 14-14 Plot of residuals versus predicted values.

14-4 General Factorial Experiments

Model for a **three-factor factorial experiment**

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases}$$

Table 14-9 Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Squares	F_0
A	SS_A	$a-1$	MS_A	$\sigma^2 + \frac{bcn\sum\tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn\sum\beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn\sum\gamma_k^2}{c-1}$	$\frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum(\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$\frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an\sum\sum(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$\frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum(\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$\frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

14-4 General Factorial Experiments

Example 14-2

A mechanical engineer is studying the surface roughness of a part produced in a metal-cutting operation. Three factors, feed rate (A), depth of cut (B), and tool angle (C), are of interest. All three factors have been assigned two levels, and two replicates of a factorial design are run. The coded data are shown in Table 14-10.

Table 14-10 Coded Surface Roughness Data for Example 14-2

	Depth of Cut (B)				
	0.025 inch		0.040 inch		
	Tool Angle (C)		Tool Angle (C)		
Feed Rate (A)	15°	25°	15°	25°	$y_{i\cdots}$
	9	11	9	10	
20 inches per minute	7	10	11	8	75
	10	10	12	16	
30 inches per minute	12	13	15	14	102

R commands and outputs

Example 14-2: enter data by row

```
> Roughness=c(9,11,9,10, 7,10,11,8, 10,10,12,16, 12,13,15,14)
> Feed=c(1,1,1,1, 1,1,1,1, 2,2,2,2, 2,2,2,2)
> Depth=c(1,1,2,2, 1,1,2,2, 1,1,2,2, 1,1,2,2)
> Angle=c(1,2,1,2, 1,2,1,2, 1,2,1,2, 1,2,1,2)
> g=lm(Roughness ~ Feed*Depth*Angle)
> anova(g)
Response: Roughness
                  Df Sum Sq Mean Sq F value   Pr(>F)
Feed               1 45.562  45.562 18.6923 0.002534 **
Depth              1 10.562  10.562  4.3333 0.070931 .
Angle              1  3.062   3.062  1.2564 0.294849
Feed:Depth         1  7.562   7.562  3.1026 0.116197
Feed:Angle         1  0.062   0.062  0.0256 0.876749
Depth:Angle        1  1.562   1.562  0.6410 0.446463
Feed:Depth:Angle   1  5.062   5.062  2.0769 0.187512
Residuals          8 19.500   2.438

> par(mfrow=c(1,3)) #
> interaction.plot(Feed, Depth, Roughness)
> interaction.plot(Feed, Angle, Roughness)
> interaction.plot(Angle, Depth, Roughness)
```

14-4 General Factorial Experiments

Example 14-2

The F-ratios for all three main effects and the interactions are formed by dividing the mean square for the effect of interest by the error mean square. Since the experimenter has selected $\alpha = 0.05$, the critical value for each of these F-ratios is $f_{0.05,1,8} = 5.32$. Alternately, we could use the P-value approach. The P-values for all the test statistics are shown in the last column of Table 14-11. Inspection of these P-values is revealing. There is a strong main effect of feed rate, since the F-ratio is well into the critical region. However, there is some indication of an effect due to the depth of cut, since $P = 0.0710$ is not much greater than $\alpha = 0.05$. The next largest effect is the AB or feed rate $\times$ depth of cut interaction. Most likely, both feed rate and depth of cut are important process variables.