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OPTIMAL PERSISTENCE POLICIES

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This paper deals with the problem of whether or not a search activity should be started and, if started, whether or not it should be continued. This problem suggests a model that is described. The model gives rise to a general functional equation for which existence and uniqueness conditions are given. Several examples are discussed, solutions to the specific functional equations appropriate to the examples are given, and the optimal policies are characterized.

This paper deals with the analysis of a special class of multistage decision problems that will be referred to as persistence problems. Research on multistage decision theory, which has progressed rapidly in the last few decades, promises an ever widening range of application. Through the work of Wald,⁵ Bellman,⁶ Arrow, Harris, and Marschak,¹¹ Shapley,⁴ and others, this field has been richly developed in a variety of applied and theoretical contexts. Study of the persistence problem may serve to extend this work in a somewhat different and perhaps useful direction.

The persistence problem is the problem of determining whether or not a search activity should be started and, if it is started, how long it should be continued. For illustration consider the investor who is interested in the possibility of buying a house. In the first place he must decide whether or not to search, or 'hunt,' for a house at all. If he decides to 'hunt' he may eventually find a house that is a reasonably good buy. The question then arises, should he settle for this one or continue to search for another, better prospect? If he continues the search there will be additional costs in terms of time, effort, and possible lost return on the immediate investment. He may also lose the option of purchasing the reasonably good buy at hand. Thus, in order to determine when the search should be stopped he must somehow weigh the possibility of finding a more suitable investment against the cost and risks of further search.

In Section 1 the persistence problem is formalized into a mathematical model and a general functional equation generated by the model is described. Results on the existence and uniqueness of solutions to this type of equa-

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tion are described in Section 2, and the formal statements and proofs are presented in the Appendix. These results are applied in Section 3 where a number of examples are described.

The persistence model was formulated originally for use as a model for certain psychological processes. Specifically, it was designed to help assess the different factors affecting behavior in situations requiring search activity. For comparison purposes, it is then of considerable interest to know the theoretical optimal behavior in these situations. However, because of the high frequency with which persistence problems are encountered in practice, the results presented here may also be of use in operations research. The mathematical results on existence and uniqueness are quite general and may be applicable to similar or related problems.

1. THE MODEL

Suppose that the search process progresses through a series of stages 0, 1, 2, \ldots. At stage \( n \) the decision-maker observes the value of a state variable \( x_n \), some point in an arbitrary state space \( S \). At each stage there is a choice between searching until the next stage or stopping at that one. If at any stage \( n \) the decision to stop is made, the process terminates and the decision-maker receives or expects the amount \( g(x_n) \). The function \( g \) defined on \( S \) will be referred to as the criterion function. For \( n = 0 \), 'stopping' is interpreted simply as not searching at all. Hence, for the initial state \( x_0 \), \( g(x_0) \) is the expected return without search. The value of the state variable \( x_n \) must summarize all the pertinent information in the actual physical situation, but as will be evident in the examples there will frequently be several alternative definitions for \( x_n \) each of which describes the same situation. The choice between equivalent variables is customarily dictated by mathematical convenience.

Let the expected cost of continuing from stage \( n \) to \( n+1 \) be \( c(x_n) \) when at \( n \) the state is \( x_n \). The function \( c \) will be called the cost function.

Given that the process is in state \( x_n \) and the choice is to continue the search, \( x_n \) is transformed stochastically at stage \( n+1 \) to a new point \( x_{n+1} \in S \) in accordance with a transition probability measure \( P_{x_n} \). That is, the probability of \( x_{n+1} \) falling into a subset \( A \) of \( S \) given \( x_n \) is \( P_{x_n}(A) \). The collection \( \{ P_x \} \) is a family of stationary Markov transition probability measures in that \( P_x \) does not depend on the integer \( n \), nor on the history of the process preceding stage \( n \), but only on the value of the state variable at \( n \), i.e., \( x_n \). Of the various assumptions which have been imposed on the model this last one, the independence of \( P_x \) from \( n \), is the most stringent. It eliminates from consideration a wide class of problems, but hopefully it still includes many. Frequently, in situations where \( P_x \) is not independent of \( n \) the information available is so complex that it is impossible
to handle analytically. The optimal solution to this simpler model may approximate the optimal solution to the general model.

Let \( V(x) \) be the expected return to the decision-maker, given that he is in state \( x \) and he chooses optimally between stopping and proceeding further. \( V(x) \) can be assumed to be independent of \( n \) because \( g, c, \) and \( P_z \) are independent of \( n \). Since the process is Markov, the optimal strategy at a given state will depend only on the state and not on the time at which the state is reached. By use of the ‘principle of optimality’ a functional equation for \( V \) may easily be written. Let

\[
G(x) = \int S V(y) \, dP_z(y) - c(x). \tag{1}
\]

\( G(x) \) is the optimal future expected return when the decision at \( x \) is to proceed at least one more stage. Since the optimal choice at \( x \) must correspond to the larger of \( g(x) \) and \( G(x) \), \( V(x) \) satisfies the equation

\[
V(x) = \max \{ g(x), G(x) \}. \tag{2}
\]

The determination of \( V \) is accomplished by an analysis of equation (2). Once \( V \) has been found the optimal policy is readily described. The space \( S \) is divided into two regions, \( g^* \) and \( G^* \), where \( g^* = \{ x | g(x) \geq G(x) \} \) and \( G^* = \{ x | g(x) \leq G(x) \} \). For \( x \) in \( G^* \) but not in \( g^* \) the optimal decision is to continue. This decision is maintained until the state variable is transformed to a point in \( g^* \). For points in \( g^* \) and \( G^* \) (i.e., boundary points) it is immaterial whether the search is stopped or not, but for points in \( g^* \) but not in \( G^* \) the decision is to stop.

To illustrate these concepts consider the hypothetical ‘house-hunting’ situation mentioned above. At \( n = 0 \) the investor has the option of not searching or attempting to buy a house at all. This alternative has value \( z_0 \). If he decides to search, and continues until stage \( n \), he will have found and inspected \( n \) houses, any of which he may buy if he wishes. The value to him of buying the \( i \)th house is \( z_i \). Since he searches at random in a large population of houses, \( z_1, z_2, \ldots, z_n \) form a sequence of independent, identically distributed random variables with a common distribution function \( F \). If the investor stops he may either not buy a house, which choice still has value \( z_0 \), or buy any of the houses he has seen. The value of stopping is then the value of the best of these possibilities. Let \( x_n = \max \{ z_0, z_1, \ldots, z_n \} \). The state space is \( S = [x_0, \infty) \) and \( g(x_n) = x_n \). The decision to continue to the next stage has a certain expected cost, which in this case will be assumed constant and greater than zero; i.e., \( c(x_n) = c > 0 \). The constant \( c \) can be interpreted as the average cost of finding and inspecting another house. The collection of probability measures governing the transitions \( x_n \rightarrow x_{n+1} \) is given by
\[ P_x(A) = \begin{cases} 
F(x) + \int_{A_x} dF(y) & \text{for } x \in A \\
\int_{A_x} dF(y) & \text{for } x \notin A
\end{cases} \] (3)

where \( A_x = A \cap (x, \infty) \).

Substitution of these definitions in (2) produces the equation

\[ V(x) = \max \left\{ x, V(x) F(x) + \int_x^\infty V(y) dF(y) - c \right\}. \quad (x \geq x_0) \] (4)

For the case in which \( F \) is continuous and strictly increasing, one solution to equation (4) is

\[ V(x) = \max \{ x, x^* \}, \quad (x \geq x_0) \] (5)

where \( x^* \) is uniquely defined by

\[ x^* = \left[ \int_{x^*}^\infty y dF(y) - c \right] / \int_{x^*}^\infty dF(y). \] (6)

If there were another solution to (4) besides (5), it would be necessary to determine which of the two competing solutions corresponded to the optimal expected return. However, an argument based on the results of the next section proves that (5) is the unique solution of (4). Hence, equation (5) must give the optimal expected return. Accordingly \( G^* \) and \( g^* \) are given by \( G^* = \{ x | x \leq x^* \} \) and \( g^* = \{ x | x \geq x^* \} \). If \( x_0 > x^* \), \( x_0 \in g^* \) and no search is undertaken at all. If \( x_0 < x^* \), search is continued until the first house of value exceeding \( x^* \) is found, which is then purchased.

Equation (6) can be written in another way:

\[ c \ E(n^*) = E(y | y > x^*) - x^* \] (7)

where \( E(n^*) \) is the expected number of stages required to find a house of value greater than \( x^* \). If \( x^* \) is to be the boundary, or indifference point, between stopping and searching further, then at this point the expected additional gain from persisting until \( x \) exceeds \( x^* \) is just offset by the expected additional cost this would incur.

2. EXISTENCE AND UNIQUENESS

The theorems that will be proved in the Appendix are devoted to an analysis of the family of solutions to the functional equation

\[ V(x) = \max \left\{ g(x), \int_S V(y) dP_x(y) - c(x) \right\}. \quad (x \in S) \] (8)

Formally, this equation is a special case of a general stochastic dynamic programming equation cited by Bellman \(^2\) (p. 124). However, because
of the different interpretation attached to the quantities entering into the equation, the conditions and consequently the proofs of the theorems differ from those of Bellman.

Once the functional equation (2) or (8) has been determined (i.e., $g$, $e$, and $P_x$ are determined), two questions arise naturally. Is there a solution to the equation, and if so, is there more than one? The latter question is usually the more important one. Often by manipulation and/or guess-work it is possible to find a solution to the equation which you feel represents the sought-after optimal strategy to the problem. This conclusion will be valid if in fact there are no other solutions to the equation. Consequently, the question of uniqueness becomes vital.

The existence of at least one solution is usually easy to verify, and a general existence theorem (Theorem 1) is given in the Appendix. Uniqueness is more difficult to establish and in numerous instances does not hold. Witness the following simple game—*Double Plus or Nothing*:

The player can decide not to play for which he receives nothing, or at a cost of one dollar he can enter the game. Upon entry a fair coin is tossed. If the outcome of the toss is heads, the player is paid $2^i + e_i$ dollars, but if the outcome is tails, he is paid nothing (i.e., he loses his original dollar) and the game ends. Should the player win on the first round he has the option of leaving the game with his winnings or risking all his winnings for a chance to play the second round. At the second round a toss of heads pays $2^2 + e_2$ dollars, but a tail pays nothing and the game ends. Similarly, at round $n$ heads pays $2^n + e_n$ dollars but tails loses everything. The $e_n$ form a bounded, strictly increasing sequence of positive values.

The discrete version of the functional equation (8), i.e.,

$$V_i = \max \{ g_i, \sum_j p_{ij} V_j - c_i \} \quad (i \geq 0) \quad (9)$$

can be applied to this game in an attempt to determine the optimal strategy. Let

$$g_i = \begin{cases} 0 & \text{for } i = 0, \\ 2^i + e_i & \text{for } i \geq 1; \end{cases}$$

$$c_i = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } i \geq 1; \end{cases}$$

$$p_{ij} = \begin{cases} \frac{1}{2} & \text{for } j = 0, \quad i \geq 0, \\ \frac{1}{2} & \text{for } j = i + 1, \quad i \geq 0. \end{cases}$$

The values $\{V_i\}$ corresponding to an optimal strategy must certainly satisfy (9) with the definitions (10). However, an analysis of this equation readily discloses that any sequence of the form
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\[ V_i = \begin{cases} 
\alpha & \text{for } i = 0, \\
2^i + \alpha & \text{for } i \geq 1,
\end{cases} \]

with \( \alpha \geq \varepsilon = \lim_{n \to \infty} \varepsilon_n \) constitutes a solution. Hence, there is an infinity of solutions.

The example serves not only as an illustration of nonuniqueness but also as an instance in which the functional equation approach is inappropriate. In this multitude of solutions the one which seems to correspond to the real game has \( \alpha = \varepsilon \). But in actuality this value is unattainable and the corresponding strategy it produces is unreasonable. At each stage the decision for this solution is 'go on.' But this strategy leads to ruin with probability one.

The main uniqueness theorem (Theorem 2) in the Appendix requires that the space \( S \) be closed and bounded and that the family of transition probability functions \( P_z \) spread their mass over the whole space \( S \). Precisely, for any open set \( 0 \subset S, P_z(0) \) must be positive. In the event \( S \) consists of a finite collection of discrete points each point is interpreted as an open set. Subject to these conditions, the theorem proves there is at most one solution \( V(x) \) to (8) which is continuous and satisfies \( \max_{x \in S} V(x) \leq \max_{x \in S} g(x) \). The last restriction is an obvious one; the expected return cannot be higher than the highest value of the criterion function. For a collection of discrete points every function is interpreted as continuous.

Corollaries 1 and 2 partially relax the conditions of Theorem 2 in order to treat cases where from some states transitions to certain subsets of \( S \) have probability zero. Briefly, Corollary 1 proves that if there is a subset \( S' \) of \( S \) on which it is known that \( V(x) = f(x) \), a known function, then, provided the complement of \( S' \) is closed and bounded and \( S' \) can be reached from all \( x \in S \) [i.e., \( P_x(S') > 0 \) for all \( x \in S \)], there is at most one solution. Corollary 2 removes the closedness and boundedness of \( S - S' \) and the continuity properties imposed in Corollary 1.

3. EXAMPLES

Random-Selection Problems

A number of problems give rise to the equation

\[ V(x) = \max \left\{ x, V(x) F(x) + \int_{x+}^\infty V(y) \, dF(y) - c \right\} \]

(11)

described in Section 2. The state variable which generates this equation is \( x_n = \max \{ z_0, z_1, \ldots, z_n \} \) where the \( z_i, i \geq 1 \), are independently, identically distributed according to \( F(z) \). In the house buying problem, \( z_i \) (\( i \geq 1 \))
is the value of the i\textsuperscript{th} house inspected. However, by interpreting the \( z_i \) broadly different models are obtained.

1. Let \( z_i = \sqrt{u_i v_i} \), where \((u_i, v_i)\) are positive and independently, identically distributed according to \( H(u,v) \). To illustrate, suppose a house has one value, \( u \), for a husband and another value, \( v \), for his wife. For reasons suggested by the work of Nash and Zeuthen (see reference 3) the couple adopts as the joint value of a house the quantity \( \sqrt{uv} \). The distribution \( F(z) \) can be derived from \( H \) by standard techniques.

2. Let \( z_i = au_i + bw_i \), \( a, b \geq 0 \), \( a + b = 1 \). The weights \( a \) and \( b \) are measures of relative accuracy of judgment of the joint value to the husband and wife, \( u \) being the judgment of the husband, \( v \) being the judgment of the wife, \( z \) being the judgment to be used in the actual decision.

These examples generalize immediately to the following:

3. Let \( z_i = \phi(u_{i1}, \ldots, u_{is}) \) where the s-tuples \((u_{i1}, \ldots, u_{is})\) are independently, identically distributed according to \( H(u_1, \ldots, u_s) \) and \( \phi \) is an arbitrary function. The \( u_i \) may be understood to correspond to different attributes of a multidimensional object and the function \( \phi \) to a weighted evaluation of an object with attributes \( u_1, \ldots, u_s \). Thus in the case of a house, \( u_1 \) might be the price, \( u_2 \) the location, \( u_3 \) the amount of dry rot, etc.

The basic random-selection problem can be generalized to multiple selection as is illustrated by the following example. Suppose an organization desires to hire \( r \) persons on the basis of a performance predictor \( u \) whose distribution is \( H(u) \) in the large population of prospective employees. A single prospect can be considered (i.e., located, interviewed, tested, etc.) at a cost of \( c \). The expected worth to the organization of a prospect who scores \( u \) is estimated to be \( v = \phi(u) \). A procedure is to be used in which prospects are considered one at a time. The process can be stopped at any time, and when it is stopped any number not exceeding \( r \) of the prospects considered up to that time may be hired subject to the condition their worth is positive. The state variable at a given stage is a vector whose coordinates are the ordered values of the \( r \) largest quantities \( z_i = \max(0, v_i) \) corresponding to the prospects available at that stage. For the initial stage, \( n = 0 \), the state variable has value \((0,0, \ldots, 0)\). Accordingly, take \( S \) to be the space of all ordered nonnegative \( r \)-tuples, \( \{(x_1, \ldots, x_r) | 0 \leq x_1 \leq \cdots \leq x_r\} \). The criterion function for stopping is defined to be \( \sum_i x_i \).

The distribution \( F \) of \( z = \max(0, v) \) may be determined from \( H \). If at any stage it is decided to continue and the new prospect has over-all value \( y = \max(0, v) \), then the state variable \((x_1, \ldots, x_r)\) is transformed to \((x_1, \ldots, x_r)\) if \( y \leq x_1 \), to \((y, x_2, \ldots, x_r)\) if \( x_1 < y \leq x_2 \), to \((x_2, y, x_3, \ldots, x_r)\) if \( x_2 < y \leq x_3 \), \ldots, and to \((x_2, \ldots, x_r, y)\) if \( x_i < y \). Therefore, the functional equa-
tion for multiple selection is

\[
V(x_1, \ldots, x_r) = \max \left\{ \sum_{i=1}^r x_i, V(x_1, \ldots, x_r) F(x_1) 
\right.
\]

\[
\left. + \int_{x_1+}^{x_2} V(y, x_2, \ldots, x_r) \, dF(y) + \int_{x_2+}^{x_3} V(x_2, y, \ldots, x_r) \, dF(y) + \ldots + \int_{x_{r-1}+}^{x_r} V(x_2, \ldots, y) \, dF(y) - c \right\}. \tag{12}
\]

The distribution \( F \) may be arbitrary except for being bounded, i.e., there exists an \( x' < \infty \) for which \( F(x') = 1 \). This assumption is not restrictive since \( x' \) may be arbitrarily large.

A solution to (12) is

\[
V(x_1, \ldots, x_r) =
\begin{cases}
\sum_{i=1}^r x_i & \text{for } x_1, \ldots, x_r < x^* \\
(r-1) \int_{x'}^{x^*} V(y, x_2, \ldots, x_r) \, dF(y) - c \quad 1 - F(x'^{-}) + x_r & \text{for } x_1, \ldots, x_{r-1} < x^* \leq x_r \\
\vdots & \vdots \\
\sum_{i=1}^r x_i & \text{for } x^* \leq x_1, \ldots, x_r
\end{cases}
\tag{13}
\]

where

\[
x^* = \inf \left\{ x : x \geq \int_{x'}^{x^*} V(y, x_2, \ldots, x_r) \, dF(y) - c \right\} \quad 1 - F(x^{'-}) \tag{14}
\]

This solution must correspond to the optimal expected return since it is the only solution to (12) which satisfies \( V(x_1, \ldots, x_r) \leq M = r x' \) for \( x_r \leq x' \).

Uniqueness of (13) will be established for the case in which \( F \) is continuous at \( x' \); the proof when \( F(x') > 0 \) is similar.

For any \( V \) satisfying \( V(x_1, \ldots, x_r) \leq M \) for \( x_r \leq x' \), as \( x_1 \to x' \)

\[
\int_{x_1+}^{x_2} V(y, x_2, \ldots, x_r) \, dF(y) + \ldots + \int_{x_{r-1}+}^{x_r} V(x_2, \ldots, y) \, dF(y) \to 0
\]

uniformly in \( x_3, \ldots, x_r \). Therefore, there exists a constant \( m < x' \) such that if \( V \) is a solution to (12) then \( V(x_1, \ldots, x_r) = \sum_{i=1}^r x_i \) for \( (x_1, \ldots, x_r) \) in the region

\[
S_0 = \{(x_1, \ldots, x_r) | m < x_1 \leq \ldots \leq x_r \}.
\]

For fixed \( x_2, \ldots, x_r \) satisfying \( m < x_2 \leq \ldots \leq x_r \), view (12) as an equation in the variable \( x_1 \) alone, i.e.,

\[
V_{x_2, \ldots, x_r}(x_1) = \max \left\{ x_1 + \sum_{i=2}^r x_i, V_{x_2, \ldots, x_r}(x_1) F(x_1) 
\right.
\]

\[
\left. + \int_{x_1}^{m} V_{x_2, \ldots, x_r}(y) \, dF(y) + \int_{m}^{x'} \left( y + \sum_{i=2}^r x_i \right) \, dF(y) - c \right\}. \tag{15}
\]
With the identifications
\[ S' = (m, x'), \quad g(x) = x + \sum x_i, \]
\[ f(x) = x + \sum x_i, \quad (x \in S') \]
\[ V(x) = V_{x_2, \ldots, x_r}(x). \]

Corollary 2 applied to (15) proves that \( V(x_1, \ldots, x_r) \) is also unique in the region
\[ S_1 = \{(x_1, \ldots, x_r) | 0 \leq x_1 \leq x_2; m < x_2 \leq \cdots \leq x_r \}. \]

For fixed \( x_3, \ldots, x_r \), satisfying \( m < x_3 \leq \cdots \leq x_r \), view (12) as an equation in the variables \( x_1, x_2 \) alone, i.e.,
\[
V_{x_3, \ldots, x_r}(x_1, x_2) = \max \left\{ x_1 + x_2 + \sum \limits_{i=3}^{r} x_i, \; V_{x_3, \ldots, x_r}(x_1, x_2) \; F(x_1) \right. \\
+ \int_{x_1^+}^{x_2} V_{x_3, \ldots, x_r}(y, x_2) \; dF(y) + \int_{x_2^+}^{m} V_{x_3, \ldots, x_r}(x_2, y) \; dF(y) \\
+ \int_{m}^{x_2} V(x_3, \ldots, x_r) \; dF(y) + \cdots + \int_{x_r^+}^{x'} V(x_3, \ldots, x_r, y) \; dF(y) - c \right\}.
\]

Corollary 2 with
\[ S' = \{(x_1, x_2) | x_2 > m \}, \quad g(x_1, x_2) = x_1 + x_2 + \sum \limits_{i=3}^{r} x_i, \]
\[ f(x_1, x_2) = V(x_1, x_2, \ldots, x_r), \quad [(x_1, x_2) \in S'] \]
\[ V(x_1, x_2) = V_{x_2, \ldots, x_r}(x_1, x_2), \]
implies \( V(x_1, \ldots, x_r) \) is unique in the region
\[ S_2 = \{(x_1, \ldots, x_r) | 0 \leq x_1 \leq x_2 \leq x_3; m < x_2 \leq \cdots \leq x_r \}. \]

\( S_0 \subset S_1 \subset S_2 \). Repetition of this argument \( r-2 \) additional steps extends the proof of uniqueness to the region \( S_r = S \).

For the case \( r = 1 \) the solution (13) reduces to the solution cited in Section 2 with \( x_0 = 0 \). Addition to the problem of the initial constant \( x_0 > 0 \) produces the solution (5). The above uniqueness proof is of course valid for \( r = 1 \), but in this special case the proof can be modified so as to replace the boundedness condition on \( F \) by the less stringent assumption \( \int_0^{x_2} y \; dF(y) < \infty \).

The decision regions \( g^* \) and \( G^* \) for the multiple problem are easily described:
\[ g^* = \{(x_1, \ldots, x_r) | x^*_1 \leq x_1 \leq \cdots \leq x_r \}, \quad G^* = S - g^*. \]
i.e., if \( x^* \geq 0 \), the search is continued until \( r \) prospects have been located with value equal or exceeding the critical point \( x^* \), or if \( x^* < 0 \), no search is undertaken at all.

In the above discussion, complete option has always been assumed. Suppose instead the decision maker is faced with the possibility of losing any or every option if he does not exercise it at the time the prospect is first encountered. The optimal policy for this problem is the same as in the case of complete option. The optimal strategy without loss of option requires acceptance of a prospect if its value exceeds \( x^* \). Clearly, this strategy may be used in the modified problem. Also, it achieves exactly the same expectation as is obtained with option. Since the optimal expectation cannot be greater without option, this policy has the largest expectation obtainable in either case.

\[ \begin{array}{c}
\mathbb{R} \\
\pi \\
n \\
x
\end{array} \]

Figure 1

A Parking Problem

In this problem a person wishes to find a place to park in order to attend a theater, shop, etc. Since he desires to arrive as soon as possible, he seeks a policy that will yield the minimum expected time of arrival. Suppose that he approaches the theater from an initial distance of \( \bar{x} \) units away and if he passes the theater without parking he circles adjacent blocks (of the same perimeter size \( 2\pi \) units) in search of parking places. His distance, \( d(x) \), from the theater regarded as a function of the distance traveled, \( x \), is of the form shown in Fig. 1.

Let \( W \) and \( D \) be the amounts of time required to walk and drive, respectively, a unit distance. \( W \) and \( D \) are assumed constant. Parking places occur along the route at random, i.e., they constitute a Poisson process with inhomogeneous density \( \lambda(x) \). Regardless of the number of occurrences preceding \( x \) the probability of a parking place occurring in the interval \( (x, x + \Delta x) \) is \( \lambda(x) \Delta x + o(\Delta x) \). The density function \( \lambda \) is periodic with period \( 2\pi \) for \( x \geq \bar{x} \).

The states for this problem are taken to be available parking places.
If the person reaches \( x \) and finds a parking place there, he can either park or continue driving. Should he park at \( x \) the additional time consumed in reaching the theater is just the walking time \( Wd(x) \). If he should drive on and proceed optimally thereafter, the expected additional time until arrival is

\[
\int_x^\infty (V(y) + D(y-x)) \lambda(y) \exp[-\Lambda(y) + \Lambda(x)] \, dy
\]

where \( \Lambda(z) = \int_0^z \lambda(u) \, du \) and \( V(y) \) is the minimum expected time of arrival from state \( y \). The expression \( \lambda(y) \exp[-\Lambda(y) + \Lambda(x)] \, dy \) is just the probability that the first parking place after \( x \) occurs between \( y \) and \( y + dy \), and \( D(y-x) \) is the driving time to \( y \). By the principle of optimality

\[
V(x) = \min \left\{ Wd(x), \int_x^\infty (V(y) + D(y-x)) \lambda(y) \right. \\
\left. \quad \cdot \exp[-\Lambda(y) + \Lambda(x)] \, dy \right\}.
\]  

(17)

Note that this formulation assumes the driver cannot see ahead to any advantage.

For ease in exposition consider first the purely circular parking problem, i.e., \( x \geq \bar{x} \). Let \( \theta = x - \bar{x} \) and let \( \text{Arg}(\theta) \) be the principal value of \( \theta \) measured from zero, i.e., \( \theta = n \cdot 2\pi + \text{Arg}(\theta) \), \( 0 \leq \text{Arg}(\theta) < 2\pi \), \( n = 0, 1, \ldots \). The state space \( S \) is the space of \( \text{Arg}(\theta) \), i.e., the interval \( [0, 2\pi] \) with the points 0 and \( 2\pi \) identified. Let

\[
\bar{d}(\theta) = \begin{cases} \\
\theta & \text{for } 0 \leq \theta \leq \pi \\
2\pi - \theta & \text{for } \pi \leq \theta \leq 2\pi,
\end{cases}
\]

\[
\bar{\lambda}(\theta) = \lambda(\bar{x} + \theta), \quad \bar{\Lambda}(\theta) = \int_0^\theta \bar{\lambda}(u) \, du.
\]

With these identifications the reduced form of (17) becomes

\[
\bar{V}(\theta) = \min \left\{ W \bar{d}(\theta), \int_0^\infty \left\{ \bar{V}[\text{Arg}(\theta')] + D(\theta' - \theta) \right\} \bar{\lambda}(\theta') \right. \\
\left. \quad \cdot \exp[-\bar{\Lambda}(\theta') + \bar{\Lambda}(\theta)] \, d\theta' \right\}
\]  

(18)

for \( 0 \leq \theta \leq 2\pi \).

Two cases must be distinguished in presenting the solution to (18). If
$$W \pi < \{ 1 - \exp(-\bar{\Lambda}(2\pi)) \}^{-1}$$

\[
\begin{align*}
&\left\{ \int_{2\pi}^{2\pi} W(2\pi-\theta') \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta') + \bar{\Lambda}(\pi)] \, d\theta' \\
&+ \int_{2\pi}^{3\pi} W(\theta'-2\pi) \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta') + \bar{\Lambda}(\pi)] \, d\theta' \\
&+ D \int_{\pi}^{\infty} (\theta'-\pi) \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta') + \bar{\Lambda}(\pi)] \, d\theta',
\end{align*}
\]

(19)

the solution to (18) is the degenerate one

$$\tilde{V}(\theta) = \bar{d}(\theta), \quad (0 \leq \theta \leq 2\pi)$$

(20)

i.e., the driver accepts the first available parking space. If, on the other hand, condition (19) holds with < replaced by \(\geq\), the solution is

$$\tilde{V}(\theta) = \begin{cases} 
W \theta & \text{for} \quad 0 \leq \theta \leq \theta^*, \\
W \theta^* - D (\theta - \theta^*) & \text{for} \quad \theta^* \leq \theta \leq \theta^{**}, \\
W (2\pi - \theta) & \text{for} \quad \theta^{**} \leq \theta \leq 2\pi,
\end{cases}$$

(21)

where

$$\theta^{**} = [2\pi W - \theta^* (W + D)] / (W - D)$$

(22)

and \(\theta^*\) is the unique value of \(\theta\) between \(\theta = 2\pi D / (W + D)\) and \(\pi\) which satisfies

\[
\begin{align*}
&\int_{\theta^*}^{\theta^{**}} [-D \theta' + \theta^* (W + D)] \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta')] \, d\theta' \\
&+ \int_{\theta^{**}}^{\theta + \theta^*} [2\pi - \theta'] \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta')] \, d\theta' \\
&+ \int_{\theta + \theta^*}^{2\pi} W [\theta' - 2\pi] \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta')] \, d\theta' \\
&= \{ 1 - \exp(-\bar{\Lambda}(2\pi)) \} \int_{\theta^*}^{\infty} [-D \theta' + \theta^* (W + D)] \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta')] \, d\theta'.
\end{align*}
\]

(23)

Furthermore, this solution is unique. The function \(\tilde{V}\) is shown in Fig. 2.

The proof that this function is a solution to (18) depends upon two supplementary propositions. First,

$$W \theta < \{ 1 - \exp(-\bar{\Lambda}(2\pi)) \}^{-1} \left\{ \int_{\theta}^{2\pi} [W \theta - D(\theta' - \theta)] \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta')] \\
+ \bar{\Lambda}(\theta)] \, d\theta' + \int_{2\pi}^{2\pi + \theta} W [\theta' - 2\pi] \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta') + \bar{\Lambda}(\theta)] \, d\theta' \\
+ D \int_{\theta}^{\infty} (\theta' + \theta) \bar{\Lambda}(\theta') \exp[-\bar{\Lambda}(\theta') + \bar{\Lambda}(\theta)] \, d\theta',
\right\}$$

(24)
which can be checked by a simple manipulation of terms. Secondly, the expression

\[
W_{\theta^*} - \left\{1 - \exp[-\Lambda(2\pi)]\right\}^{-1} \left\{ \int_{\theta^*}^{\theta^{**}} [W_{\theta^*} - D (\theta' - \theta^*)] \Lambda(\theta') \exp[-\Lambda(\theta')] + \Lambda(\theta^*) \right\} d\theta' + \int_{\theta^*}^{2\pi} W [2\pi - \theta'] \Lambda(\theta') \exp[-\Lambda(\theta') + \Lambda(\theta^*)] d\theta' + \int_{2\pi}^{2\pi + \theta^*} W [\theta' - 2\pi] \Lambda(\theta') \exp[-\Lambda(\theta') + \Lambda(\theta^*)] d\theta' + D \int_{\theta^*}^{\infty} (\theta' - \theta^*) \Lambda(\theta') \exp[-\Lambda(\theta') + \Lambda(\theta^*)] d\theta'
\]

(25)

regarded as a function of \(\theta^*\) has at most one zero in the interval \([\theta, \pi]\); a differentiation argument will verify this.

The proof of uniqueness is straightforward. Any solution of (18) must be continuous since \(W \tilde{d}(\theta)\) is a continuous function of \(\theta\) and both

\[
\int_{\theta}^{\infty} \tilde{V}[\text{Arg}(\theta')] \Lambda(\theta') \exp[-\Lambda(\theta') + \Lambda(\theta)] d\theta'
\]

and

\[
D \int_{\theta}^{\infty} (\theta' - \theta) \Lambda(\theta') \exp[-\Lambda(\theta') + \Lambda(\theta)] d\theta'
\]

are continuous by the Lebesgue convergence theorem. But Theorem 2, or rather the min analog of this theorem, with \(S = [0, 2\pi]\), where 0 and \(2\pi\) are identified, proves that (20) or (21) is the unique continuous solution of (18).

The full parking problem can now be handled easily. For \(x \geq \bar{x}\) the solution to (17) is \(V(x) = \tilde{V}[\text{Arg}(x - \bar{x})]\). For \(x < \bar{x}\) there exists a point \(x^*\) to which the person drives without parking and then accepts the first available space between \(x^*\) and \(\bar{x}\) if one occurs. Altogether
Optimal Persistence Policies

\[ V(x) = \begin{cases} 
  D(x^* - x) + W \, d(x^*) & \text{for } 0 \leq x \leq x^* \\
  W \, d(x) & \text{for } x^* \leq x \leq \bar{x} \\
  \bar{V}[\text{Arg}(x - \bar{x})] & \text{for } \bar{x} \leq x.
\end{cases} \quad (26) \]

The point \( x^* \) is the equilibrium point at which the value of stopping and proceeding are exactly equal; i.e.,

\[
W \, d(x^*) = \int_{x^*}^{\bar{x}} W \, d(y) \, \lambda(y) \exp[-\Lambda(y) + \Lambda(x^*)] \, dy
\]

\[
+ \int_{x^*}^{\infty} \bar{V}[\text{Arg}(y - \bar{x})] \, \lambda(y) \exp[-\Lambda(y) + \Lambda(x^*)] \, dy
\]

\[
+ D \int_{x^*}^{\infty} (y - x^*) \, \lambda(y) \exp[-\Lambda(y) + \Lambda(x^*)] \, dy.
\]

The function (26) is the unique solution to (17). Uniqueness has already been established for the region \( x \geq \bar{x} \), and this can be extended to the whole region by an application of Corollary 1 with \( S' = (\bar{x}, \infty) \) and \( f(x) = \bar{V}[\text{Arg}(x - \bar{x})] \).

In the special case of a homogeneous Poisson process in the interval \( x \geq \bar{x} \), i.e., \( \lambda(\theta) = \lambda > 0 \) for all \( \theta \), the various expressions naturally simplify. In particular, condition (19) reduces to

\[
\lambda < -\frac{1}{\pi} \log[2W/(W+D) - 1],
\]

and \( \theta^* \) and \( \theta^{**} \) may be determined from the equations

\[
\int_0^T \lambda \exp[-\lambda \theta] \, d\theta = (2W/W+D) \int_0^S \lambda \exp[-\lambda \theta] \, d\theta,
\]

\[
T \, (W+D) = 2WS - 2\pi D,
\]

where \( \theta^* = T - S \) and \( \theta^{**} = 2\pi - S \).

A Quiz-Show Problem

The quiz program furnishes a problem of some psychological interest as a judgment situation that can be analyzed in terms of the finite, discrete version of equation (8). Let there be \( n+1 \) different levels or states, i.e., let \( S = \{0,1,\ldots,n\} \). Let \( g_i \) be the amount the contestant receives if he reaches level \( i \) and decides to quit. For simplicity let \( g_n \) be the largest of the \( g_i \). The state 0 will correspond to the state of having lost and being removed from the contest.

For the contestant who has reached state \( i \geq 1 \) there is a probability \( p_{ij} \) that he will move to state \( j \) on the next round if he continues. As in the more complicated quiz programs these probabilities may be positive
for states other than the next higher state or the state of having lost. The probability of losing, \( p_{i0} \), will be assumed to be positive for all \( i \). For \( i=0, \ p_{00}=1 \) since the state 0 permits no further progress. The cost of continuing when in state \( i \) is \( c_i \). For all \( i, \ c_i \geq 0 \).

If \( V_i \) is the optimal expected winnings for a contestant at level \( i \), \( (V_0, V_1, \cdots, V_n) \) must satisfy the discrete form of equation (8); that is,

\[
V_i = \max \{ g_i, \sum_j p_{ij} V_j - c_i \}. \quad (i=0, 1, \cdots, n)
\]

By Theorem 1 this system of equations has at least one solution which satisfies \( \max_i \{ V_i \} = V_n = g_n \). Furthermore, by Corollary 2 there is only one solution for which \( V_0 = g_0, \ \max_i \{ V_i \} = V_n = g_n \). For \( S' \) in Corollary 2 choose the set consisting of state 0, and let \( f_0 = g_0 \). Consequently, there is no problem as to the existence and uniqueness of the solution; the only difficulty occurs in finding it. However, at least for \( n \leq 3 \) there is an easy graphical method.

Let \( n=3 \). Then \( V_3 = g_3, \ V_0 = g_0 \). If the constants \( V_1 \) and \( V_2 \) are to constitute the remainder of the solution, they must satisfy the system of inequalities

\[
c_1 - p_{10} g_0 - p_{12} g_2 \geq (p_{11} - 1) V_1 + p_{12} V_2, \quad V_1 \geq g_1, \quad (30)
\]

\[
c_2 - p_{20} g_0 - p_{23} g_3 \geq p_{21} V_1 + (p_{22} - 1) V_2, \quad V_2 \geq g_2, \quad (31)
\]

where in both (30) and (31) at least one of the inequalities must be equality. In the \( (x_1, x_2) \)-plane construct the lines

\[
l_1: \ (p_{11} - 1)x_1 + p_{12} x_2 = c_1 - p_{10} g_0 - p_{12} g_3, \quad (32)
\]

\[
l_2: \ p_{21} x_1 + (p_{22} - 1)x_2 = c_2 - p_{20} g_0 - p_{23} g_3, \quad (33)
\]
and plot the point \( g = (g_1, g_2) \) as shown in Fig. 3. The solution point \( V = (V_1, V_2) \) is obtained by moving from the point \( g = (g_1, g_2) \) to the shaded region \( R \) in the manner indicated. The region \( R \) consists of all \( (x_1, x_2) \) that satisfy

\[
(p_{11} - 1) \, x_1 + p_{12} \, x_2 \leq c_1 - p_{10} \, g_0 - p_{13} \, g_3,
\]

\[
p_{21} \, x_1 + (p_{22} - 1) \, x_2 \leq c_2 - p_{20} \, g_0 - p_{23} \, g_3.
\]

For \( n > 3 \), this technique involves traveling toward or along hyperplanes in \((n-1)\)-dimensional space.

We are indebted to Professor David Blackwell for suggesting the form of the functional equation to be used in treating the problems described above.

**APPENDIX**

The following assumptions will be implicit in the conditions of the theorems. \( S \) is an arbitrary space and \( \mathcal{A} \) is a \( \sigma \)-algebra of measurable subsets of \( S \), i.e., \((S, \mathcal{A})\) is a measurable space. The function \( g \) is measurable on \( S \), and the function \( c \) is nonnegative and measurable on \( S \). \( \{P_x\} \) is a collection of probability measures defined on \( \mathcal{A} \). In the event \( S \) is also assumed to be a topological space, \( \mathcal{A} \) will be assumed to be the class of Baire sets. The functions \( g \) and \( c \) will then be assumed continuous, and \( \{P_x\} \) becomes a collection of Baire measures.

The definitions, theorems, and proofs are stated abstractly for generality, but the essence is still retained if \( S \) is taken to be a subset of the real line and

\[
\int V(y) \, dP_x(y)
\]

is replaced by \( \int V(y) \, p_x(y) \, dy \) in the density case and by \( \sum_i V(y_i) \, P_x(y_i) \) in the discrete case. Continuity of a function then means continuity in the ordinary sense, and the terms \( \sigma \)-algebra, measurable, topological, and Baire may be overlooked.

**Theorem 1.** If \( g \) is bounded on \( S \), then equation (8) has at least one solution \( V \) which satisfies \( \sup_{x \in S} V(x) \leq \sup_{x \in S} g(x) \).

**Proof.** Consider the problem truncated after \( N \) steps, and define

\[
\begin{align*}
V^0(x) &= g(x), \\
V^1(x) &= \max \{g(x), \int_S V^0(y) \, dP_x(y) - c(x)\}, \\
&\quad \cdots \\
V^N(x) &= \max \{g(x), \int_S V^{N-1}(y) \, dP_x(y) - c(x)\}.
\end{align*}
\]  

(34)

\( V^N(x) \leq V^{N+1}(x) \), and since \( V^N(x) \leq \sup_{x \in S} g(x) \) for all \( N \) and \( x \), \( \lim_{N \to \infty} V^N(x) = V^*(x) \) exists and is measurable. Passage to the limit on both sides of (34) yields

\[
V^*(x) = \max \{g(x), \lim_{N \to \infty} \int_S V^{N-1}(y) \, dP_x(y) - c(x)\}
\]

\[
= \max \{g(x), \int_S V^*(y) \, dP_x(y) - c(x)\},
\]
where the second equality holds by the Lebesgue convergence theorem. Hence, $V^*$ is a solution and $\sup_{x \in S} V^*(x) \leq \sup_{x \in S} g(x)$.

**Theorem 2.** Let $S$ be a compact topological space, and for $x \in S$ let $P_x(0) > 0$ hold for all open sets $0 \subseteq S$. Then, equation (8) has at most one solution $V$ which satisfies (i) $V$ is continuous, (ii) $\max_{x \in S} V(x) = \max_{x \in S} g(x)$.

**Proof.** For a function $R(x)$ on $S$ let $\gamma_r(x) = \int_S R(y) \, dP_x(y) - R(x)$. If $R$ is a solution of (8), then $\gamma_r(x) \leq c(x)$ for all $x \in S$.

Suppose $V$ and $U$ are two solutions of (8) which satisfy (i) and (ii). Let

$$S_1 = \{ x | V(x) = U(x) \}, \quad S_2 = \{ x | V(x) > U(x) \}, \quad S_3 = \{ x | V(x) < U(x) \}.$$

The proof consists in showing that $S_2 = S_3 = \emptyset$, the null set.

Suppose $S_2 \neq \emptyset$. Define $W(x) = \min \{ V(x), U(x) \}$. For $x \in S_1 \cup S_3$, $W(x) = V(x)$, and for all $x \in S$, $\int_S W(y) \, dP_x(y) \leq \int_S V(y) \, dP_x(y)$. Hence, for $x \in S_1 \cup S_3$, $\gamma_w(x) \leq \gamma_r(x) \leq c(x)$. Since $U(x) > g(x)$ implies $\gamma_u(x) = c(x)$, $\gamma_u(x) = c(x)$ for $x \in S_3$. Therefore, for $x \in S_3$, $\gamma_u(x) \leq \gamma_w(x)$.

On the other hand, there exists at least one point $x^0 \in S_3$ for which $\gamma_w(x^0) > \gamma_u(x^0)$. For let $x^0$ be a point in $S_3$ at which

$$W(x^0) - U(x^0) = \min_{x \in S} \{ W(x) - U(x) \} < 0. \quad (35)$$

Equation (35) implies

$$W(x^0) - U(x^0) < \int_S [W(y) - U(y)] \, dP_x^0(y) \quad (36)$$

with strict inequality since $S_1 \cup S_2 = \emptyset$ violates condition (ii). But (36) is equivalent to the assertion $\gamma_u(x^0) < \gamma_w(x^0)$. Thus, the assumption $S_3 \neq \emptyset$ leads to a contradiction.

The argument is symmetrical in $U$ and $V$ so $S_2 \neq \emptyset$ is impossible.

**Corollary 1.** Let $S$ be a topological space, and let $S'$ be a nonnull subset of $S$ for which the closure of $S - S'$ is compact and $P_x(0) > 0$ for all open sets $0 \subseteq S'$ and all $x \in S$. Let the function $f$ be continuous and satisfy $f(x) \geq g(x)$ for $x \in S'$. Then, equation (8) has at most a single solution $V$ which satisfies (i) $V$ is continuous, (ii) $V(x) = f(x)$ for $x \in S'$.

**Proof.** Assume $S_3 \neq \emptyset$. As in the proof of Theorem 2, $\gamma_w(x) \leq \gamma_u(x)$ for $x \in S_3$. In contradiction to this let $x^0 \in S_3$ be defined by

$$W(x^0) - U(x^0) = \min_{x \in S} \{ W(x) - U(x) \} = \min_{x \in (S - S')} \{ W(x) - U(x) \}. \quad (37)$$

From (37),

$$W(x^0) - U(x^0) < \int_S [W(y) - U(y)] \, dP_x^0(y) \quad (38)$$

with strict inequality since $S' \subset S_1$ and $P_x^0(0) > 0$ for all open sets $0 \subseteq S'$.

**Corollary 2.** Let $S$ be a measurable space and $S'$ be a nonnull measurable subset of $S$ for which $P_x(S') \geq \delta > 0$ for all $x \in S'$ and $g$ is bounded on $S - S'$. Let the function $f$ be measurable and satisfy $f(x) \geq g(x)$ for $x \in S'$. Then, equation (8) has at most a single measurable solution $V$ which satisfies (i) $V$ is bounded on $S - S'$, (ii) $V(x) = f(x)$ for $x \in S'$.
Proof. Suppose $V$ and $U$ are two measurable solutions, and suppose $S_3 \neq \emptyset$. By the same argument as in the proof of Theorem 2, $\gamma_u(x) \leq \gamma_u(x)$ for $x \in S_3$.

Let $-M = \inf_{x \in S} \{W(x) - U(x)\}$, $0 < M < \infty$. For any fixed $\epsilon < M \delta (1 + \delta)^{-1}$, let $x^*$ be a point in $S_3$ for which $W(x^*) - U(x^*) < M + \epsilon$, and let

$$S^* = \{x \mid [W(x^*) - U(x^*)] - [W(x) - U(x)] > 0\}.$$  

Then

$$\int_{S} \{[W(x^*) - U(x^*)] - [W(y) - U(y)]\} \, dP_{x^*}(y) \leq \epsilon P_{x^*}(S^*) + (-M + \epsilon) P_x(S^*)$$

$$+ \int_{S-(S^* \cup S^*)} \{[W(x^*) - U(x^*)] - [W(y) - U(y)]\} \, dP_{x^*}(y). \quad (39)$$  

Since the integrand over $S-(S^* \cup S^*)$ is nonpositive and $\epsilon + (-M + \epsilon) \delta < 0$, equation (39) implies

$$W(x^*) - U(x^*) < \int_{S} [W(y) - U(y)] \, dP_{x^*}(y), \quad (40)$$

or, equivalently, $\gamma_u(x^*) > \gamma_u(x^*)$ which is a contradiction.

Corollaries 1 and 2 relax the strict compactness imposed on the space $S$ in Theorem 2. Counterexamples like Double Plus or Nothing indicate that any further, significant relaxation of the compactness only results in nonuniqueness. Uniqueness does not even extend to the case of noncompact $S$ but bounded criterion values as another example shows. Let

$$S = \{0, 1, 2, \ldots\};$$

$$g_i = \begin{cases} \frac{1}{2} & \text{for } i = 0, \\ (\frac{1}{2})^{n+1} & \text{for } i = 2n - 1, \, n \geq 1, \\ 1 - (\frac{1}{2})^{n+1} & \text{for } i = 2n, \, n \geq 1; \end{cases} \quad (41)$$

$$c_i = \begin{cases} \frac{1}{2} & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases}$$

The transition probabilities $\{p_{ij}\}$ in (41) can be defined so that both

$$V_{i}' = \begin{cases} \frac{3}{4} & \text{for } i = 0 \\ 1 & \text{for } i \geq 1, \end{cases}$$

and $V_{i}'' = g_i$ for all $i$, constitute solutions. Namely, for $i = 0$, let $p_{00} = \frac{1}{2}$ and $p_{0j}, \, j \geq 1$, be arbitrary so long as $\sum_j p_{0j} g_j \leq \frac{3}{4}$, and for $i > 0$ let $p_{i0} = 0$ and $p_{ij}, \, j \geq 1$, be arbitrary so long as $g_i \geq \sum_j p_{ij} g_j$.

The above example could be extended to an example with a finite number of $c_i > 0$; additional balancing of $\{V_i\}$ and $\{p_{ij}\}$ is all that is required. Uniqueness can be established if the sequence of criterion values is increasing and bounded and the $\{p_{ij}\}$ are nondegenerate, so apparently the oscillatory character of the above values is essential for nonuniqueness.

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