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Department of Statistics

Statistics 19 **Instructor: Nicolas Christou**

Portfolio risk and return

Mean and variance of the return of a stock:
Closing prices (Figure 1) show how the IBM stock fluctuates from January 2000 to December 2005. We can mention here the high volatility (variance) that is exhibited in stocks. Let us define the return at time t of a stock as follows:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

where P_t, P_{t-1} are the closing stock prices at time t and $t-1$ respectively. One can use daily, weekly, or monthly returns but in portfolio management, we usually use monthly returns. The previous definition for the return of a stock is a common one to obtain returns of stocks. For example, if the stock's closing price at the beginning of last month was \$50 while at the beginning of this month it is \$51 then the return during this period is 2%. The formula for the returns can include dividends paid to the shareholders. In this case the formula becomes $R_t = \frac{P_t + D - P_{t-1}}{P_{t-1}}$, where D are the dividends paid between time t and $t-1$. In this paper, the closing prices were used, but one can use the adjusted prices. The adjusted prices adjust the price of the stock for dividends paid or stock splits. The websites *http://finance.yahoo.com* and *http://wrds.wharton.upenn.edu* provide both the closing and adjusted prices. Also, we will define the mean and the variance of the returns of stock i as

$$\bar{R}_i = \frac{1}{n}\sum_{t=1}^{n} R_{it},\ \sigma_i^2 = \frac{1}{n-1}\sum_{t=1}^{n}(R_{it} - \bar{R}_i)^2 \tag{2}$$

and the covariance between the returns of stocks i and j as

$$cov(R_i, R_j) = \sigma_{ij} = \frac{1}{n-1}\sum_{t=1}^{n}(R_{it} - \bar{R}_i)(R_{jt} - \bar{R}_j) \tag{3}$$

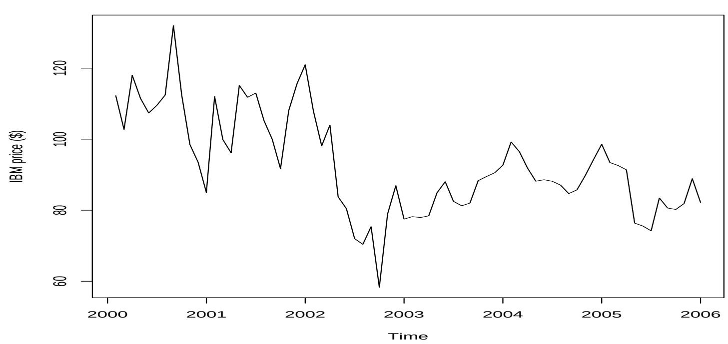


Figure 1: IBM closing price, January 2000 - December 2005.

The returns of IBM for the same period are shown on Figure 2 below:

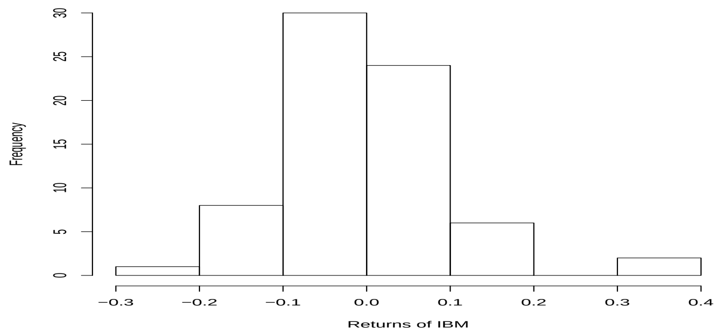


Figure 2: Returns of IBM, January 2000 - December 2005.

Similarly, for the stocks Exxon-Mobil and Boeing we obtain the plots below:

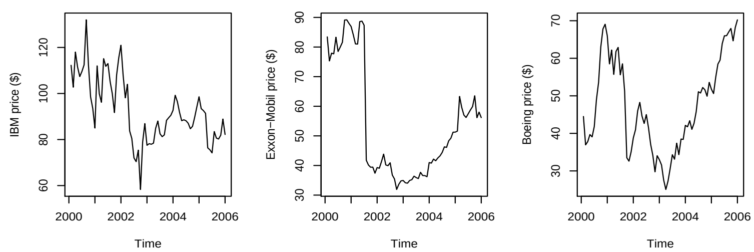


Figure 3: Closing prices of IBM, Exxon-Mobil, Boeing, January 2000 - December 2005.

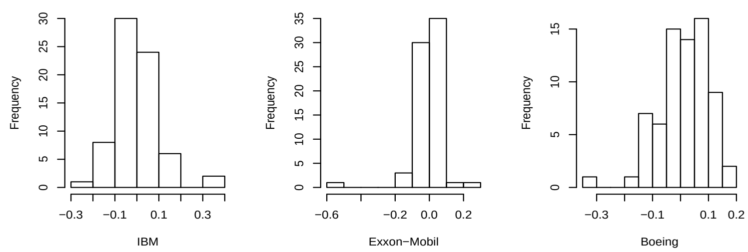


Figure 4: Returns of IBM, Exxon-Mobil, Boeing, January 2000 - December 2005.

Performance of the market:
Stock market performance is measured by some indexes. In the US the oldest is the DJIA (since 1896). Since 1928, it has consisted of the average of 30 stocks. Originally it contained 20 stocks. Today it is computed by adding the price of the 30 stocks and dividing by some adjustment factor. It is widely used but it has some flaws (30 stocks cannot represent the entire market). The next most popular index in the U.S. is the Standard and Poor's Composite (S & P 500) index. The figures below show the fluctuations of the market over several years.

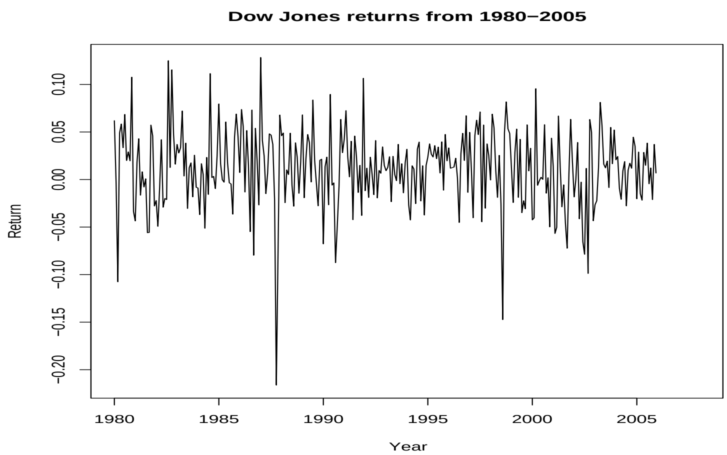


Figure 5: Returns of Dow Jones January 1980 - December 2005.

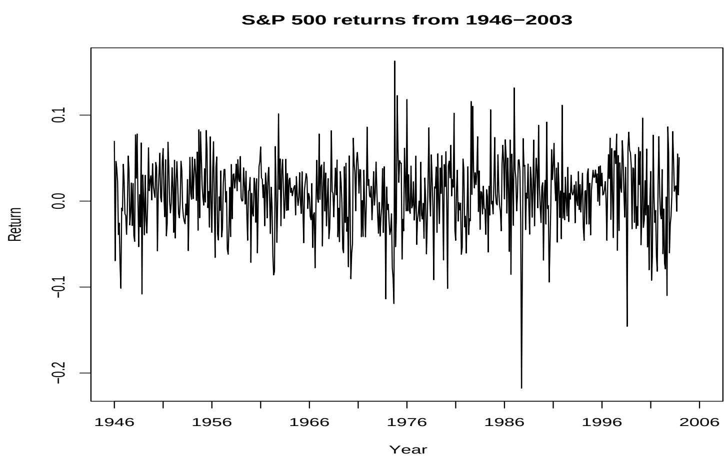


Figure 6: Returns of S&P 500January 1946 - December 2003.

“Black” Monday:
In financial markets, Black Monday refers to Monday, October 19, 1987, when stock markets around the world crashed. In the United States the Dow Jones Industrial Average (DJIA) dropped by 508 points to 1739 (-22.6%). How many standard deviations was this return away from the mean of the distribution below if we assume normality?

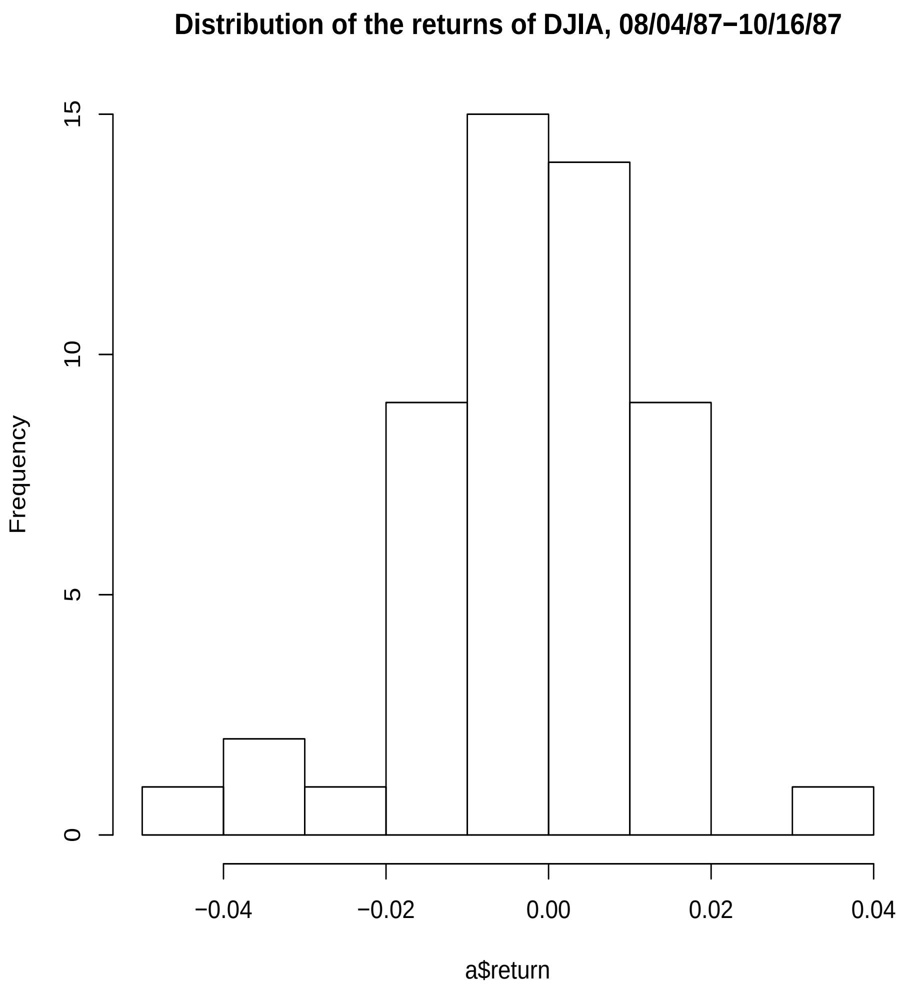


Figure 7: Daily returns of DJIA from 04 August 1987 - 15 October 1987 .

```
      Min.    1st Qu.     Median       Mean    3rd Qu.       Max.
-0.0460100 -0.0105900 -0.0008823 -0.0023040  0.0079460  0.0301800
> sd(a$djia_ret)
[1] 0.01446897
```

Investing in a portfolio:
An investor has a certain amount of dollars to invest into two stocks (IBM and $TEXACO$). A portion of the available funds will be invested into IBM (denote this portion of the funds with x_A) and the remaining funds into TEXACO (denote it with x_B) - so $x_A + x_B = 1$. The resulting portfolio will be $x_A R_A + x_B R_B$, where R_A is the monthly return of IBM and R_B is the monthly return of $TEXACO$. The goal here is to find the most efficient portfolios given a certain amount of risk. Using market data from January 1980 until February 2001 we compute that $E(R_A) = 0.010$, $E(R_B) = 0.013$, $Var(R_A) = 0.0061$, $Var(R_B) = 0.0046$, and $Cov(R_A, R_B) = 0.00062$.
We first want to minimize the variance of the portfolio. This will be:

$$\begin{aligned} &\text{Minimize } \ \mathrm{Var}(x_A R_A + x_B R_B) \\ &\qquad\quad \text{subject to } \ x_A + x_B = 1 \end{aligned}$$

Or

$$\begin{aligned} &\text{Minimize } \ x_A^2 Var(R_A) + x_B^2 Var(R_B) + 2x_A x_B Cov(R_A, R_B) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{subject to } \ x_A + x_B = 1 \end{aligned}$$

Therefore our goal is to find x_A and x_B, the percentage of the available funds that will be invested in each stock. Substituting $x_B = 1 - x_A$ into the equation of the variance we get

$$x_A^2 Var(R_A) + (1 - x_A)^2 Var(R_B) + 2x_A(1 - x_A)Cov(R_A, R_B)$$

To minimize the above exression we take the derivative with respect to x_A, set it equal to zero and solve for x_A. The result is:

$$x_A = \frac{Var(R_B) - Cov(R_A, R_B)}{Var(R_A) + Var(R_B) - 2Cov(R_A, R_B)}$$

and therefore

$$x_B = \frac{Var(R_A) - Cov(R_A, R_B)}{Var(R_A) + Var(R_B) - 2Cov(R_A, R_B)}$$

The values of x_a and x_B are:

$$x_a = \frac{0.0046 - 0.0062}{0.0061 + 0.0046 - 2(0.00062)} \Rightarrow x_A = 0.42.$$

and $x_B = 1 - x_A = 1 - 0.42 \Rightarrow x_B = 0.58$. Therefore if the investor invests 42% of the available funds into IBM and the remaining 58% into $TEXACO$ the variance of the portfolio will be minimum and equal to:

$$Var(0.42R_A + 0.58R_B) = 0.42^2(0.0061) + 0.58^2(0.0046) + 2(0.42)(0.58)(0.00062) = 0.002926$$

Therefore $Sd(0.42R_A + 0.58R_B) = \sqrt{0.002926} = 0.0541$.
The corresponding expected return of this porfolio will be:

$$E(0.42R_A + 0.58R_B) = 0.42(0.010) + 0.58(0.013) = 0.01174.$$

We can try many other combinations of x_A and x_B (but always $x_A + x_B = 1$) and compute the risk and return for each resulting portfolio. This is shown in the table below and the graph of return against risk on the other side.

x_A	x_B	Risk (σ^2)	Return	Risk (σ)
1.00	0.00	0.006100	0.01000	0.078102
0.95	0.05	0.005576	0.01015	0.074670
0.90	0.10	0.005099	0.01030	0.071404
0.85	0.15	0.004669	0.01045	0.068329
0.80	0.20	0.004286	0.01060	0.065471
0.75	0.25	0.003951	0.01075	0.062859
0.70	0.30	0.003663	0.01090	0.060526
0.65	0.35	0.003423	0.01105	0.058505
0.60	0.40	0.003230	0.01120	0.056830
0.55	0.45	0.003084	0.01135	0.055531
0.50	0.50	0.002985	0.01150	0.054635
0.42	0.58	0.002926	0.01174	0.054088
0.40	0.60	0.002930	0.01180	0.054126
0.35	0.65	0.002973	0.01195	0.054524
0.30	0.70	0.003063	0.01210	0.055348
0.25	0.75	0.003201	0.01225	0.056580
0.20	0.80	0.003386	0.01240	0.058193
0.15	0.85	0.003619	0.01255	0.060157
0.10	0.90	0.003899	0.01270	0.062439
0.05	0.95	0.004226	0.01285	0.065005
0.00	1.00	0.004600	0.01300	0.067823

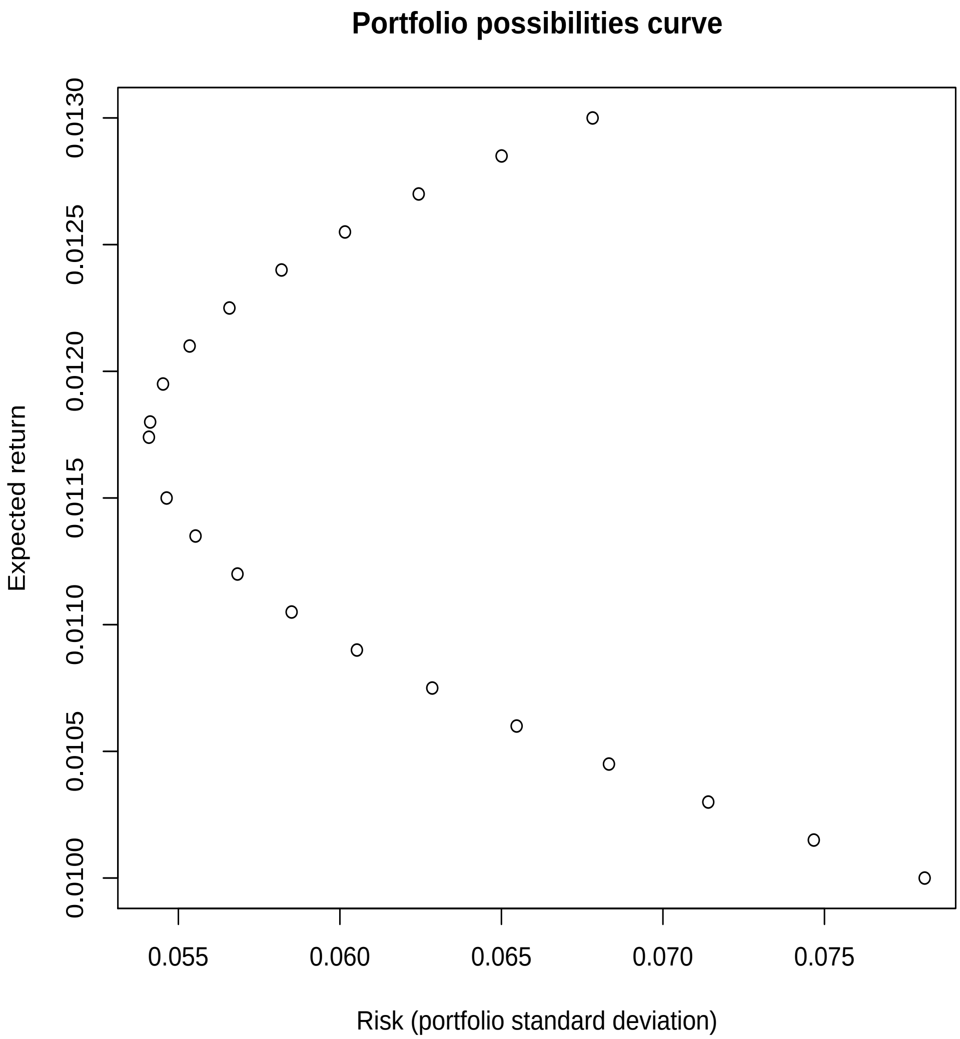


Figure 8: Portfolio possibilities curve with 2 stocks.

Combinations of two risky assets: Short sales not allowed

Define:

x_A is the fraction of available funds invested in asset A.

x_B is the fraction of available funds invested in asset B.

$\bar{R}_A$ is the expected return on the asset A.

$\bar{R}_B$ is the expected return on the asset B.

$\bar{R}_p$ is the expected return on the portfolio.

σ_A^2 is the variance of the return on asset A.

σ_B^2 is the variance of the return on asset B.

σ_{AB} $(cov(R_A, R_B))$ is the covariance between the returns on asset A and asset B.

ρ_{AB} is the correlation coefficient between the returns on asset A and asset B.

σ_p is the standard deviation of the return on the portfolio.

Correlation coefficient and the efficient frontier

The inputs of portfolio are:

- Expected return for each stock.
- Standard deviation of the return of each stock.
- Covariance between two stocks.

The correlation coefficient (ρ) between stocks A, B is always between -1, 1 and it is equal to:

$$\rho = \frac{cov(R_A, R_B)}{\sigma_A \sigma_B} \Rightarrow cov(R_A, R_B) = \rho \sigma_A \sigma_B$$

Expected return of the portfolio:

$$E(X_A R_A + X_B R_B) = X_A \bar{R}_A + X_B R_B$$

Variance of the portfolio:

$$var(X_A R_A + X_B R_B) = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2 X_A X_B \rho \sigma_A \sigma_B$$

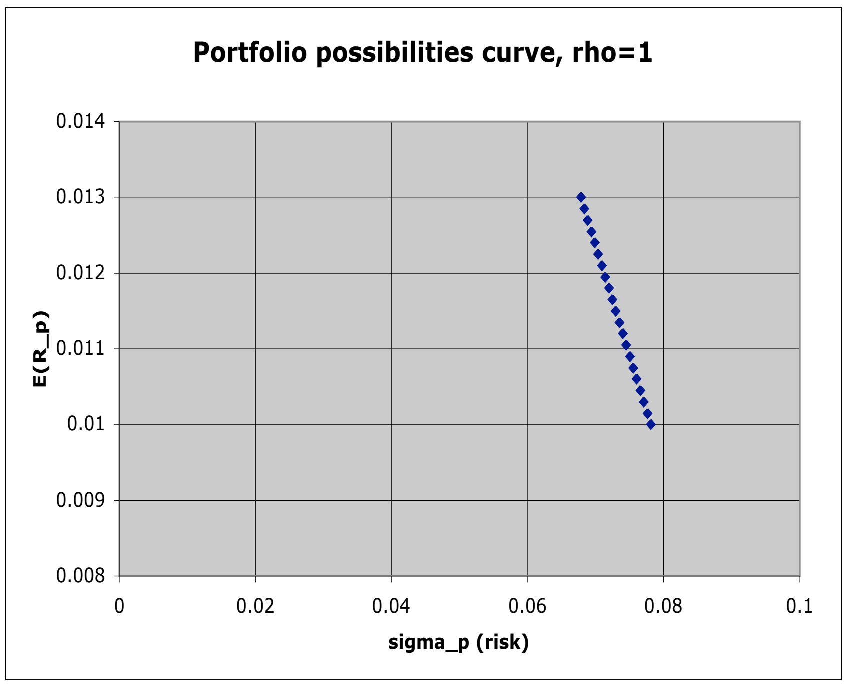
In the next pages we explore the shape of the efficient frontier for different values of the correlation coefficient.

What do you observe when $\rho = 1$, $\rho = -1$?

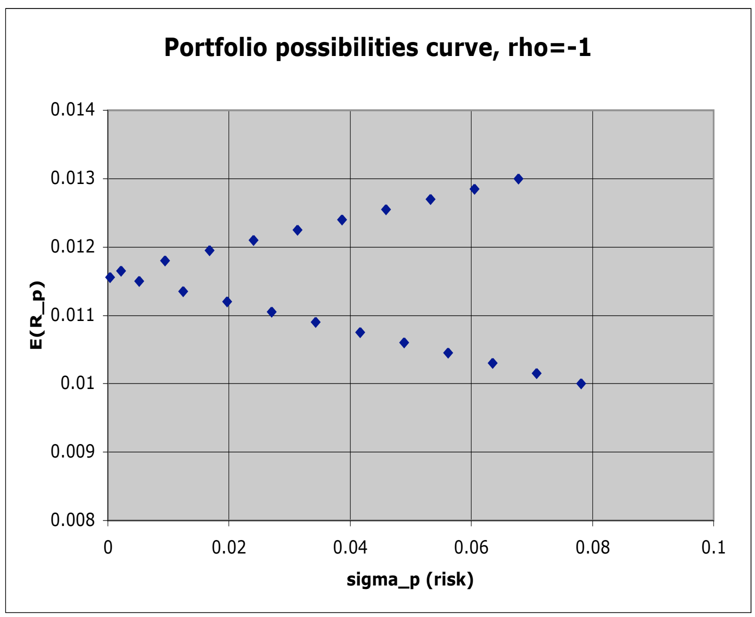
Assume
rho=1

	IBM	TEXACO
Rbar	0.01	0.013
Var	0.0061	0.0046

x1	x2	var(return)	sd(return)	E(return)
1.00	0.00	0.0061	0.0781025	0.01
0.95	0.05	0.00601998	0.07758854	0.01015
0.90	0.10	0.00594049	0.07707458	0.0103
0.85	0.15	0.00586153	0.07656062	0.01045
0.80	0.20	0.00578309	0.07604666	0.0106
0.75	0.25	0.00570519	0.0755327	0.01075
0.70	0.30	0.00562781	0.07501874	0.0109
0.65	0.35	0.00555096	0.07450478	0.01105
0.60	0.40	0.00547464	0.07399082	0.0112
0.55	0.45	0.00539885	0.07347686	0.01135
0.50	0.50	0.00532358	0.0729629	0.0115
0.45	0.55	0.00524885	0.07244894	0.01165
0.40	0.60	0.00517464	0.07193498	0.0118
0.35	0.65	0.00510096	0.07142102	0.01195
0.30	0.70	0.00502781	0.07090706	0.0121
0.25	0.75	0.00495519	0.0703931	0.01225
0.20	0.80	0.00488309	0.06987914	0.0124
0.15	0.85	0.00481153	0.06936518	0.01255
0.10	0.90	0.00474049	0.06885122	0.0127
0.05	0.95	0.00466998	0.06833726	0.01285
0.00	1.00	0.0046	0.0678233	0.013



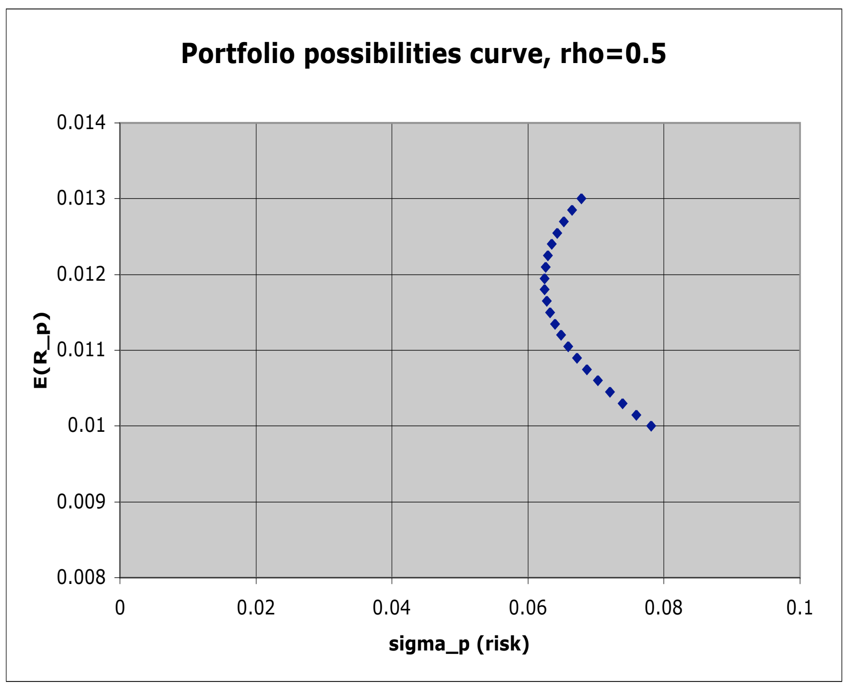
			x1	x2	var(return)	sd(return)	E(return)
Assume			1.00	0.00	0.0061	0.0781025	0.01
rho=-1			0.95	0.05	0.005013519	0.07080621	0.01015
			0.90	0.10	0.00403351	0.06350992	0.0103
			0.85	0.15	0.003159972	0.05621363	0.01045
			0.80	0.20	0.002392906	0.04891734	0.0106
			0.75	0.25	0.001732312	0.04162105	0.01075
			0.70	0.30	0.001178189	0.03432476	0.0109
	IBM	TEXACO	0.65	0.35	0.000730538	0.02702847	0.01105
Rbar	0.01	0.013	0.60	0.40	0.000389359	0.01973218	0.0112
Var	0.0061	0.0046	0.55	0.45	0.000154651	0.01243589	0.01135
			0.50	0.50	2.64155E-05	0.0051396	0.0115
			0.45	0.55	4.65132E-06	0.00215669	0.01165
			0.40	0.60	8.93589E-05	0.00945298	0.0118
			0.35	0.65	0.000280538	0.01674927	0.01195
			0.30	0.70	0.000578189	0.02404556	0.0121
			0.25	0.75	0.000982312	0.03134185	0.01225
			0.20	0.80	0.001492906	0.03863814	0.0124
			0.15	0.85	0.002109972	0.04593443	0.01255
			0.10	0.90	0.00283351	0.05323072	0.0127
			0.05	0.95	0.003663519	0.06052701	0.01285
			0.00	1.00	0.0046	0.0678233	0.013
			0.46	0.54	1.28391E-07	0.00035832	0.011555



Assume
rho=0.5

	IBM	TEXACO
Rbar	0.01	0.013
Var	0.0061	0.0046

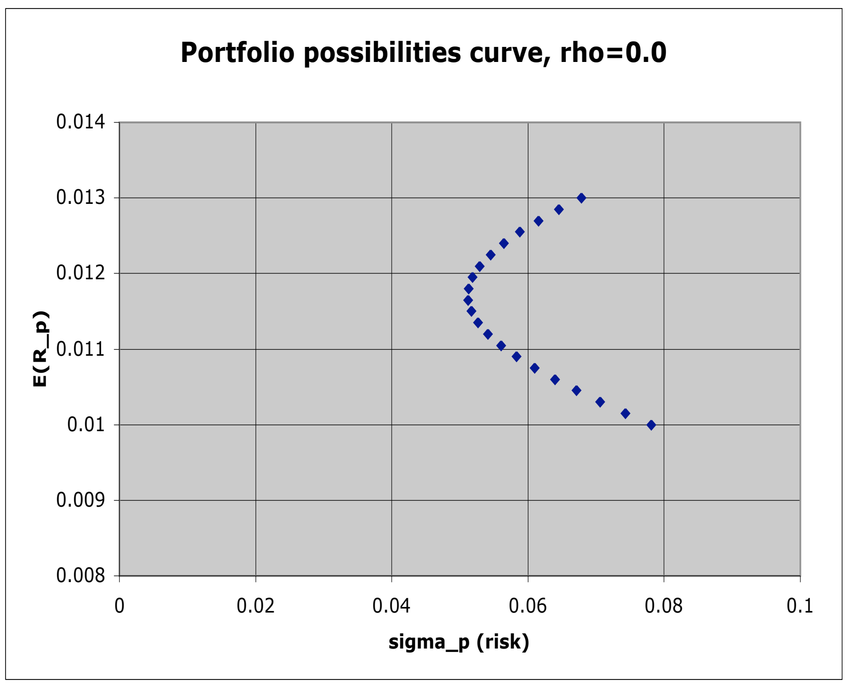
x1	x2	var(return)	sd(return)	E(return)
1.00	0.00	0.0061	0.0781025	0.01
0.95	0.05	0.00576837	0.07594976	0.01015
0.90	0.10	0.00546375	0.07391715	0.0103
0.85	0.15	0.00518614	0.07201485	0.01045
0.80	0.20	0.00493555	0.07025345	0.0106
0.75	0.25	0.00471197	0.06864378	0.01075
0.70	0.30	0.00451541	0.06719677	0.0109
0.65	0.35	0.00434586	0.06592311	0.01105
0.60	0.40	0.00420332	0.06483302	0.0112
0.55	0.45	0.0040878	0.0639359	0.01135
0.50	0.50	0.00399929	0.06323996	0.0115
0.45	0.55	0.0039378	0.06275189	0.01165
0.40	0.60	0.00390332	0.06247656	0.0118
0.35	0.65	0.00389586	0.06241679	0.01195
0.30	0.70	0.00391541	0.0625732	0.0121
0.25	0.75	0.00396197	0.06294418	0.01225
0.20	0.80	0.00403555	0.06352596	0.0124
0.15	0.85	0.00413614	0.06431282	0.01255
0.10	0.90	0.00426375	0.06529736	0.0127
0.05	0.95	0.00441837	0.06647079	0.01285
0.00	1.00	0.0046	0.0678233	0.013



Assume
rho=0.0

	IBM	TEXACO
Rbar	0.01	0.013
Var	0.0061	0.0046

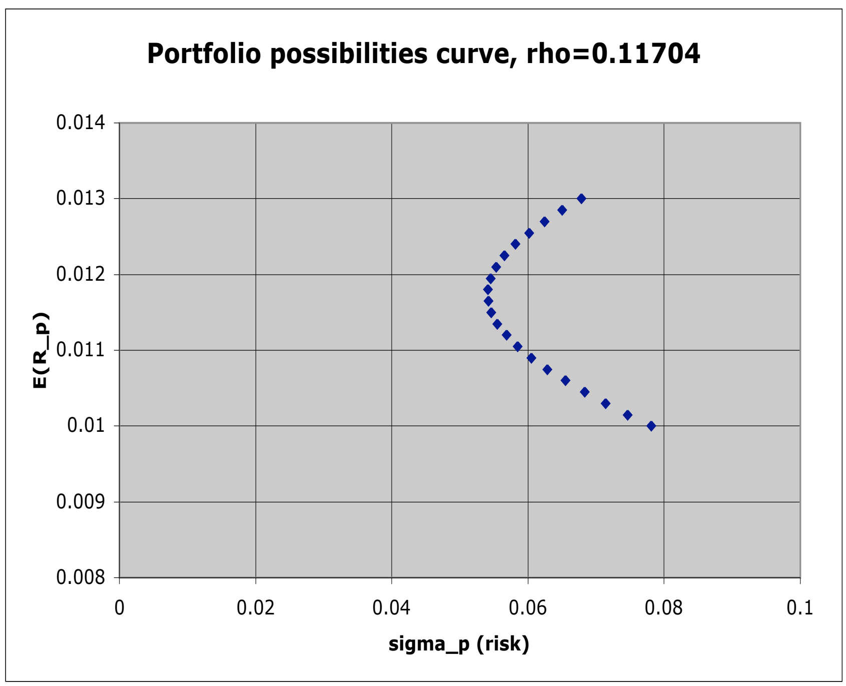
x1	x2	var(return)	sd(return)	E(return)
1.00	0.00	0.0061	0.0781025	0.01
0.95	0.05	0.00551675	0.07427483	0.01015
0.90	0.10	0.004987	0.07061869	0.0103
0.85	0.15	0.00451075	0.06716212	0.01045
0.80	0.20	0.004088	0.06393747	0.0106
0.75	0.25	0.00371875	0.06098155	0.01075
0.70	0.30	0.003403	0.05833524	0.0109
0.65	0.35	0.00314075	0.05604239	0.01105
0.60	0.40	0.002932	0.05414795	0.0112
0.55	0.45	0.00277675	0.05269488	0.01135
0.50	0.50	0.002675	0.0517204	0.0115
0.45	0.55	0.00262675	0.05125183	0.01165
0.40	0.60	0.002632	0.05130302	0.0118
0.35	0.65	0.00269075	0.05187244	0.01195
0.30	0.70	0.002803	0.05294337	0.0121
0.25	0.75	0.00296875	0.05448624	0.01225
0.20	0.80	0.003188	0.05646238	0.0124
0.15	0.85	0.00346075	0.05882814	0.01255
0.10	0.90	0.003787	0.06153861	0.0127
0.05	0.95	0.00416675	0.06455037	0.01285
0.00	1.00	0.0046	0.0678233	0.013

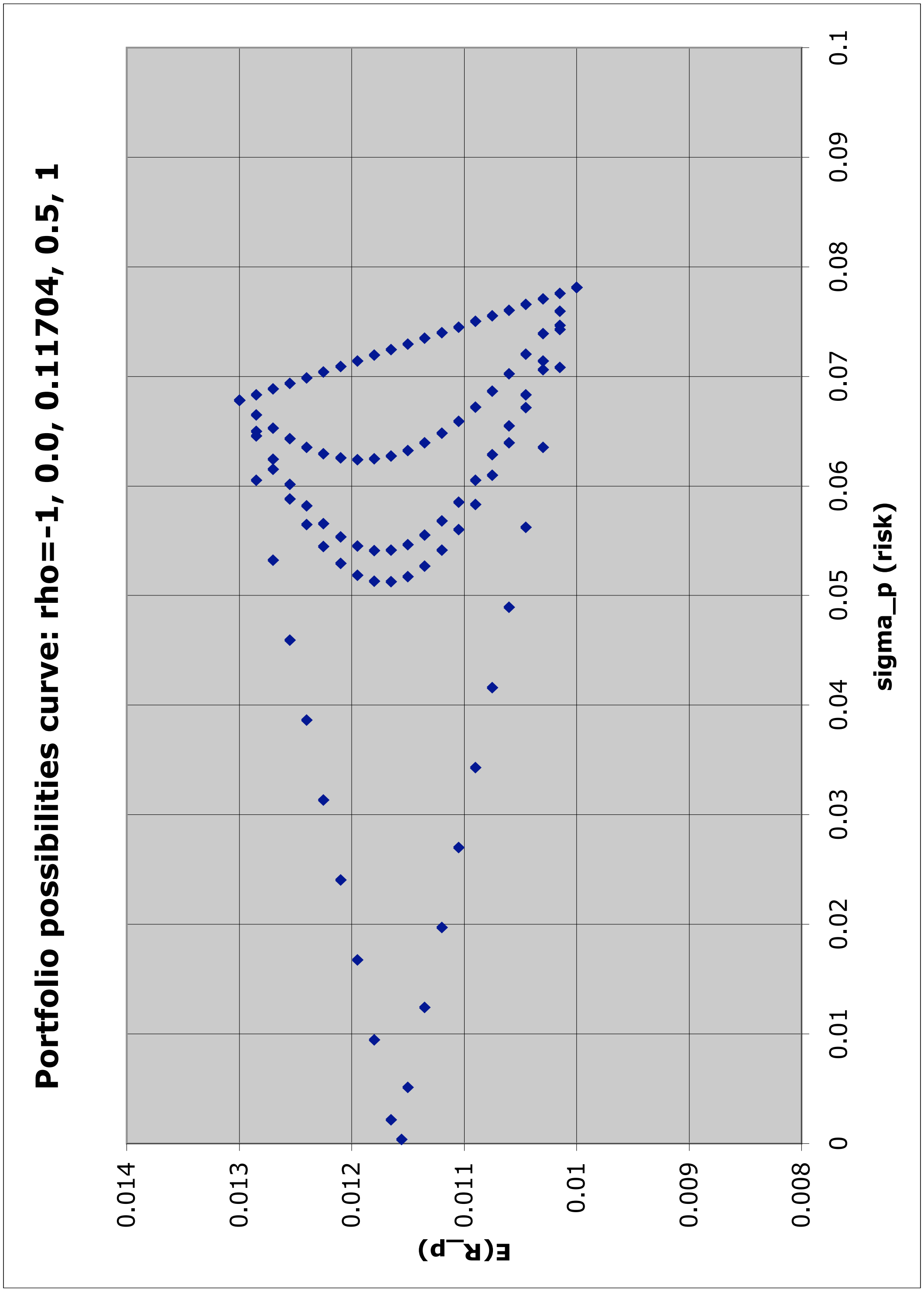


rho=0.11704

	IBM	TEXACO
Rbar	0.01	0.013
Var	0.0061	0.0046
Cov	0.00062	

x1	x2	var(return)	sd(return)	E(return)
1.00	0.00	0.0061	0.0781025	0.01
0.95	0.05	0.00557565	0.07467028	0.01015
0.90	0.10	0.0050986	0.07140448	0.0103
0.85	0.15	0.00466885	0.06832898	0.01045
0.80	0.20	0.0042864	0.0654706	0.0106
0.75	0.25	0.00395125	0.06285897	0.01075
0.70	0.30	0.0036634	0.06052603	0.0109
0.65	0.35	0.00342285	0.05850513	0.01105
0.60	0.40	0.0032296	0.05682957	0.0112
0.55	0.45	0.00308365	0.05553062	0.01135
0.50	0.50	0.002985	0.05463515	0.0115
0.45	0.55	0.00293365	0.05416318	0.01165
0.40	0.60	0.0029296	0.05412578	0.0118
0.35	0.65	0.00297285	0.05452385	0.01195
0.30	0.70	0.0030634	0.05534799	0.0121
0.25	0.75	0.00320125	0.05657959	0.01225
0.20	0.80	0.0033864	0.05819278	0.0124
0.15	0.85	0.00361885	0.06015688	0.01255
0.10	0.90	0.0038986	0.06243877	0.0127
0.05	0.95	0.00422565	0.065005	0.01285
0.00	1.00	0.0046	0.0678233	0.013




Portfolio possibilities curve: rho=-1, 0.0, 0.11704, 0.5, 1
E(R_p)
sigma_p (risk)
0.014
0.013
0.012
0.011
0.01
0.009
0.008
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Efficient frontier with three stocks:

Efficient frontier constructed using stocks IBM, TEXACO, FORD
Short sales are not allowed

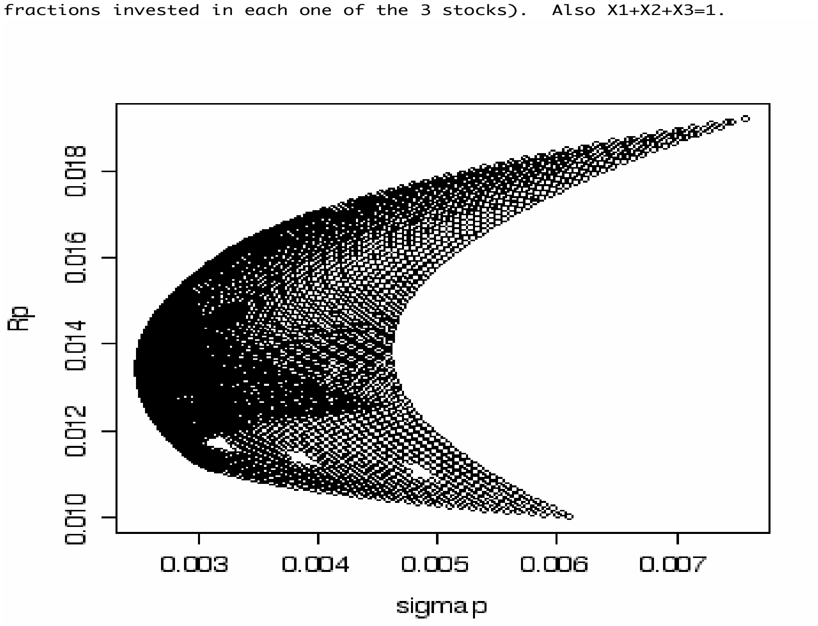
```
. sum ibm texaco ford

    Variable |      Obs        Mean    Std. Dev.       Min        Max
-------------+-----------------------------------------------------
         ibm |      253    .0099993    .0676577  -.2618315   .3176471
      texaco |      253    .0125712    .0781872  -.1944416   .3993536
        ford |      253    .0191745     .087041  -.2171053   .3372549


. correlate ibm texaco ford, cov
(obs=253)

             |      ibm   texaco     ford
-------------+---------------------------
         ibm |  .004578
      texaco |  .000619  .006113
        ford |  .002474  .000198  .007576
```

Each point on the graph below corresponds to some mean return and standard deviation of the portfolio that consists of IBM, TEXACO, FORD for some combination of X1, X2, X3 (the fractions invested in each one of the 3 stocks). Also X1+X2+X3=1.



So far ...

- The lower (closer to -1.0) the correlation coefficient between two assets the higher the benefit from diversification.
- Any combination of the two assets can never have risk more than the risk found on a straight line that connects the two assets in the expected return standard deviation space.
- We have produced a simple expression for finding the composition of the minimum risk portfolio.
- We know how to construct the portfolio possibilities curve and find the efficient frontier (concave function) for the two-asset case.

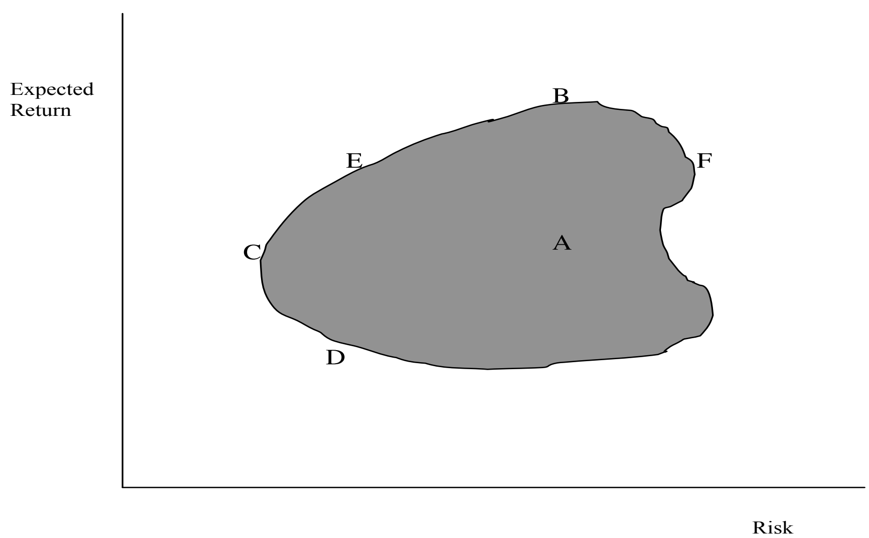
Summary:

The Efficient Frontier

In theory we could plot all risky assets and their combinations in the “Expected Return” “Risk” space to get the figure below. The investor would choose a portfolio that

1. Offers bigger expected return for the same risk, or
2. Offers a lower risk for the same expected return.

Examine portfolios A, B
C, A
D, E
F, E



What is point C?
What is point B?

Effect of diversification

Modern Portfolio Theory and Investments Analysis

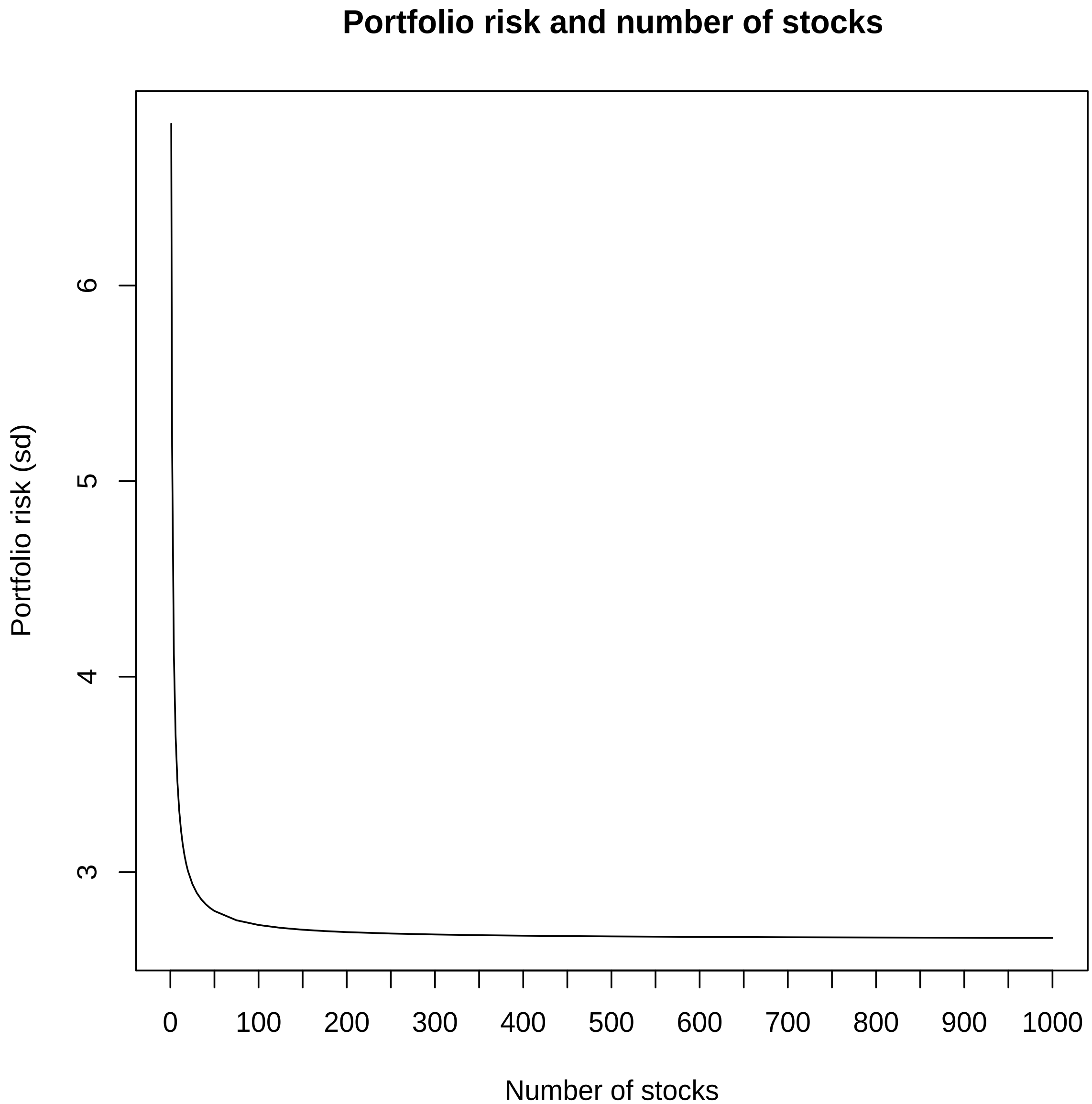
Elton, Gruber, Brown, Goetzmann, Wiley 6th Edition, 2003

The table below shows the effect of diversification when dealing with U.S. stocks. The average variance and average covariance of all stocks in the New York Exchange were computed using monthly data. The average variance was 46.619 and the average covariance was 7.058. As more and more stocks are added in the portfolio the average variance approaches the average covariance.

Number of stocks	Portfolio variance
1	46.619
2	26.838
4	16.948
6	13.651
8	12.003
10	11.014
12	10.354
14	9.883
16	9.530
18	9.256
20	9.036
25	8.640
30	8.376
35	8.188
40	8.047
45	7.937
50	7.849
75	7.585
100	7.453
125	7.374
150	7.321
175	7.284
200	7.255
250	7.216
300	7.190
350	7.171
400	7.157
450	7.146
500	7.137
600	7.124
700	7.114
800	7.107
900	7.102
1000	7.097
Infinity	7.058

See next plot . . .

The variance (risk) of a portfolio decreases as the number of stocks in the portfolio inreases:



The risk that can be diversified away it is called diversifiable risk (or unsystematic) risk, while the risk that can not be diversified away it is called non-diversifiable risk (or systematic) risk.

Simple commands using R:

```
#Read the closing prices of the three stocks:
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics_c183_c283/i
bm_xom_boeing_prices_00_05.txt", header=TRUE)

#Time plot of the closing prices of IBM:
plot(a$ibm, xaxt="n", type="l", xlab="Time", ylab="IBM price ($)")
axis(1, at=seq(0, 72, by=12),labels=seq(2000, 2006, by=1))

#Time plot of the closing prices of the three stocks: IBM, Exxon-Mobil, Boeing.
par(mfrow=c(1,3))
plot(a$ibm, xaxt="n", type="l", xlab="Time", ylab="IBM price ($)")
axis(1, at=seq(0, 72, by=12),labels=seq(2000, 2006, by=1))
plot(a$xom, xaxt="n", type="l", xlab="Time", ylab="Exxon-Mobil price ($)")
axis(1, at=seq(0, 72, by=12),labels=seq(2000, 2006, by=1))
plot(a$boeing, xaxt="n", type="l", xlab="Time", ylab="Boeing price ($)")
axis(1, at=seq(0, 72, by=12),labels=seq(2000, 2006, by=1))

#Compute the returns for the three stocks:
ribm <- (a$ibm[-1]-a$ibm[-length(a$ibm)])/a$ibm[-length(a$ibm)]
rxom <- (a$xom[-1]-a$xom[-length(a$xom)])/a$xom[-length(a$xom)]
rboeing <- (a$boeing[-1]-a$boeing[-length(a$boeing)])/a$boeing[-length(a$boeing)]

#Calculate summary statistics and variance-covariance matrix of the returns of the three stocks:
returns <- cbind(ribm,rxom,rboeing)
summary(returns)
cov(returns)

#Histogram of the returns of IBM:
hist(ribm, main="", xlab="Returns of IBM")

#Histogram of the returns of the three stocks: IBM, Exxon-Mobil, Boeing.
par(mfrow=c(1,3))
hist(ribm, main="", xlab="IBM")
hist(rxom, main="", xlab="Exxon-Mobil")
hist(rboeing, main="", xlab="Boeing")
```

Important note:
In the data set above the most recent data appear at the bottom (and the oldest on the top). Usually (at least from http://finance.yahoo.com) the Excel file that you get has the most recent data on the top of the file. In this case you will transform the prices into return in `R` **as follows:**

```
ribm <- (a$ibm[-length(a$ibm)]-a$ibm[-1])/a$ibm[-1]
rxom <- (a$xom[-length(a$xom)]-a$xom[-1])/a$xom[-1]
rboeing <- (a$boeing[-length(a$boeing)]-a$boeing[-1])/a$boeing[-1]
```