

**Short sales allowed, risk-free lending and borrowing
Point of tangency - solution**

See also, *Modern Portfolio Theory and Investments Analysis* by Elton, Gruber, Brown, Goetzmann, Wiley 6th Edition, 2003.

When the investor faces the efficient frontier and riskless lending and borrowing the combinations of the risk-free asset with the risky portfolio lie on the line:

$$\bar{R}_p = R_f + \left(\frac{\bar{R}_A - R_f}{\sigma_A} \right) \sigma_p \tag{1}$$

The solution is to find the point of tangency of this line to the efficient frontier. Let's call this point G . To find this point we want to maximize the slope of the line in (1) as follows:

$$\max \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$$

Subject to

$$\sum_{i=1}^n x_i = 1$$

Since,

$$R_f = \left(\sum_{i=1}^n x_i \right) R_f = \sum_{i=1}^n x_i R_f$$

we can write the maximization problem as

$$\max \theta = \frac{\sum_{i=1}^n x_i (\bar{R}_i - R_f)}{\left(\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij} \right)^{\frac{1}{2}}}$$

or

$$\max \theta = \left[\sum_{i=1}^n x_i (\bar{R}_i - R_f) \right] \left[\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij} \right]^{-\frac{1}{2}}$$

Take now the partial derivative with respect to each $x_i, i = 1, \dots, n$, set them equal to zero and solve. Let's find the partial derivative w.r.t. x_k :

$$\begin{aligned} \frac{\partial \theta}{\partial x_k} &= (\bar{R}_k - R_f) \left[\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij} \right]^{-\frac{1}{2}} \\ &+ \left[\sum_{i=1}^n x_i (\bar{R}_i - R_f) \right] \left[2x_k \sigma_k^2 + 2 \sum_{j=1, j \neq k}^n x_j \sigma_{kj} \right] \\ &\times \left[\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij} \right]^{-\frac{3}{2}} \times \left(-\frac{1}{2} \right) = 0 \end{aligned}$$

Multiply both sides by

$$\left[\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij} \right]^{\frac{1}{2}} \quad \text{to get}$$

$$(\bar{R}_k - R_f) - \frac{\sum_{i=1}^n x_i (\bar{R}_i - R_f)}{\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij}} (x_k \sigma_k^2 + \sum_{j=1, j \neq k}^n x_j \sigma_{kj}) = 0$$

Now, if we let

$$\lambda = \frac{\sum_{i=1}^n x_i (\bar{R}_i - R_f)}{\sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij}}$$

the previous expression will be

$$(\bar{R}_k - R_f) - \lambda x_k \sigma_k^2 - \sum_{j=1, j \neq k}^n \lambda x_j \sigma_{kj} = 0$$

or

$$\bar{R}_k - R_f = \lambda x_k \sigma_k^2 + \sum_{j=1, j \neq k}^n \lambda x_j \sigma_{kj}$$

Let's define now a new variable,

$$z_k = \lambda x_k$$

and finally

$$\bar{R}_k - R_f = z_k \sigma_k^2 + \sum_{j=1, j \neq k}^n z_j \sigma_{kj} \tag{2}$$

We have one equation like (2) for each $k = 1, \dots, n$. Here they are:

$$\begin{aligned} \bar{R}_1 - R_f &= z_1 \sigma_1^2 + z_2 \sigma_{12} + z_3 \sigma_{13} + \dots + z_n \sigma_{1n} \\ \bar{R}_2 - R_f &= z_1 \sigma_{21} + z_2 \sigma_2^2 + z_3 \sigma_{23} + \dots + z_n \sigma_{2n} \\ \dots &= \dots \\ \dots &= \dots \\ \dots &= \dots \\ \bar{R}_n - R_f &= z_1 \sigma_{n1} + z_2 \sigma_{n2} + z_3 \sigma_{n3} + \dots + z_n \sigma_n^2 \end{aligned}$$

The solution involves solving the system of these simultaneous equations, which can be written in matrix form as:

$$\mathbf{R} = \mathbf{\Sigma} \mathbf{Z}, \quad \text{where } \mathbf{\Sigma} \text{ is the variance-covariance matrix of the returns of the } n \text{ stocks.}$$

To solve for \mathbf{Z} :

$$\mathbf{Z} = \Sigma^{-1}\mathbf{R}$$

Once we find the z_i 's it is easy to find the x_i 's (the fraction of funds to be invested in each security). Earlier we defined

$$z_k = \lambda x_k \Rightarrow x_k = \frac{z_k}{\lambda}$$

We need to find λ as follows:

$$\begin{aligned} z_1 + z_2 + \cdots + z_n &= \sum_{i=1}^n z_i \\ \lambda(x_1 + x_2 + \cdots + x_n) &= \sum_{i=1}^n z_i \\ \lambda &= \sum_{i=1}^n z_i \end{aligned}$$

Therefore the composition G (the point of tangency) is given by:

$$\begin{aligned} x_1 &= \frac{z_1}{\lambda} \\ x_2 &= \frac{z_2}{\lambda} \\ x_3 &= \frac{z_3}{\lambda} \\ &\vdots \\ &\vdots \\ x_n &= \frac{z_n}{\lambda} \end{aligned}$$