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Statistics 19 **Instructor: Nicolas Christou**

Binomial option pricing model

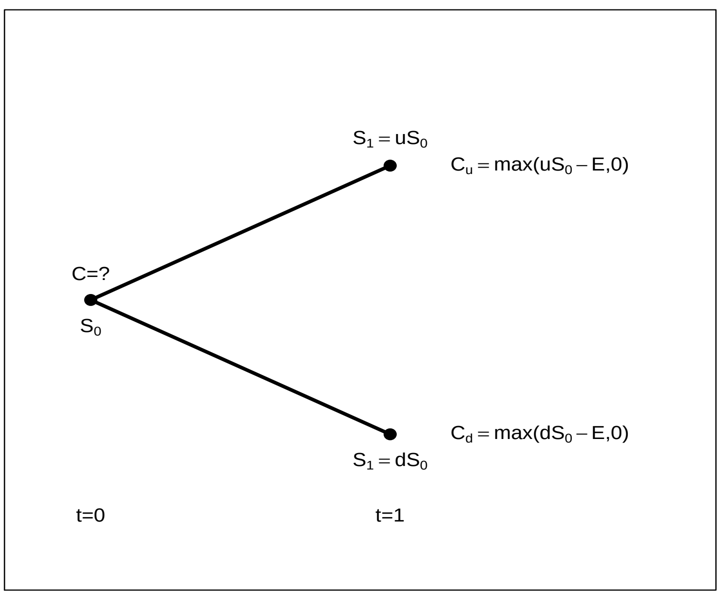
See also, *Modern Portfolio Theory and Investments Analysis* by Elton, Gruber, Brown, Goetzmann, Wiley 6th Edition, 2003.

Define:

S_0	Stock price at $t=0$
S_1	Stock price at $t=1$
E	Exercise price of the call option
u	$1+\%$ change in stock price from $t=0$ to $t=1$ if stock price increases ($u>1$) $u=e^{\sigma\sqrt{\frac{t}{n}}}$
d	$1-\%$ change in stock price from $t=0$ to $t=1$ if stock price decreases ($d<1$) $d=\frac{1}{u}$
C	The call price
α	The number of shares of stocks purchased per one call (hedge ratio)
C_u	Price of call at $t=1$ if stock price increases: $max(S_1-E,0)$ or $max(uS_0-E,0)$
C_d	Price of call at $t=1$ if stock price decreases: $max(S_1-E,0)$ or $max(dS_0-E,0)$
r	Interest rate

	Flows at $t=0$	Flows at $t=1$	
Action		$S_1=uS_0$	$S_1=dS_0$
Write 1 call	C	$-C_u$	$-C_d$
Buy α shares of stock	$-\alpha S_0$	αuS_0	αdS_0

One−step binomial tree



This will be a hedged portfolio at $t = 1$ if the payoffs at $t = 1$ are independent of the price of the stock at $t = 1$, i.e. $-C_u + \alpha u S_0 = -C_d + \alpha d S_0$. Solve for α to get $\alpha = \frac{C_u - C_d}{uS_0 - dS_0}$. The payoff at $t = 1$ is $\alpha d S_0 - C_d$, and since this is riskless payoff, the portfolio we constructed at $t = 0$ must have earned the risk free interest rate. Therefore, $(\alpha S_0 - C)(1 + r) = \alpha d S_0 - C_d$. We can solve for C to get:

$$\begin{aligned} C &= \frac{-\alpha d S_0 + C_d + (1+r)\alpha S_0}{1+r} \\ \text{Also } \alpha &= \frac{C_u - C_d}{uS_0 - dS_0} \end{aligned}$$

Using both equations we get:

$$C = \frac{C_u \frac{1+r-d}{u-d} + C_d \frac{u-(1+r)}{u-d}}{1+r}$$

Now, let $p = \frac{1+r-d}{u-d}$ and $1 - p = \frac{u-(1+r)}{u-d}$ to get

$$C = \frac{C_u p + C_d(1-p)}{1+r} \tag{1}$$

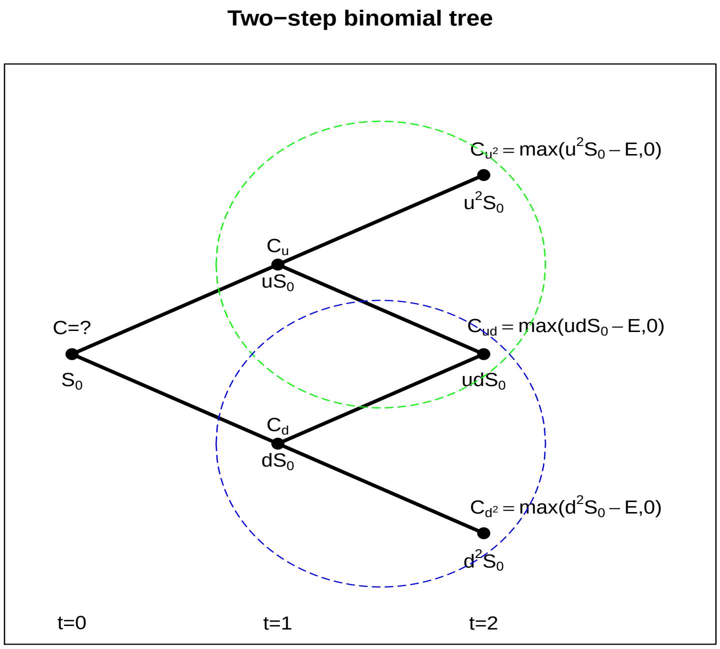
This is the price of a European call with one period to expiration.

Numerical example

Find the price of a European call if $S_0 = \$50, E = \$50, u = 1.05, d = 0.95, r = 1\%$ using one-step binomial tree. Find the price of a European put written on the same stock.

Two-step binomial tree

The previous result was for one period. The time to expiration is divided into many small intervals and the procedure above is iterated. We will examine the result for a two-step binomial tree and then generalize.



Let's examine the upper branch (top circle) of this two-step binomial tree:

	Flows at $t=1$	Flows at $t=2$	
Action		$S_2 = u^2 S_0$	$S_2 = udS_0$
Write 1 call	C_u	$-C_{u^2}$	$-C_{ud}$
Buy α shares of stock	$-\alpha u S_0$	$\alpha u^2 S_0$	αudS_0

This will be a hedged portfolio if $-C_{u^2} + \alpha u^2 S_0 = -C_{ud} + \alpha udS_0$, and solving for α we get $\alpha = \frac{C_{u^2} - C_{ud}}{u^2 S_0 - udS_0}$. Since at $t=2$ the payoff is riskless the portfolio that we constructed at time $t=1$ must have earned the risk free interest rate, i.e. $(-C_u + \alpha u S_0)(1+r) = \alpha udS_0 - C_{ud}$. Solve for C_u to get:

$$C_u = \frac{C_{u^2} p + C_{ud}(1-p)}{1+r}. \tag{2}$$

Similarly, we examine the lower branch (bottom circle) of the two-step binomial tree:

	Flows at $t=1$	Flows at $t=2$	
Action		$S_2 = udS_0$	$S_2 = d^2 S_0$
Write 1 call	C_d	$-C_{ud}$	$-C_{d^2}$
Buy α shares of stock	$-\alpha dS_0$	αudS_0	$\alpha d^2 S_0$

This will be a hedged portfolio if $-C_{ud} + \alpha udS_0 = -C_{d^2} + \alpha d^2 S_0$, and solving for α we get $\alpha = \frac{C_{ud} - C_{d^2}}{udS_0 - d^2 S_0}$. Since at $t=2$ the payoff is riskless the portfolio that we constructed at time $t=1$ must have earned the risk free interest rate, i.e. $(\alpha dS_0 - C_u)(1+r) = -C_{d^2} + \alpha d^2 S_0$. Solve for C_d to get:

$$C_d = \frac{C_{ud} p + C_{d^2}(1-p)}{1+r}. \tag{3}$$

Using equations (2) and (3) we update equation (1):

$$C = \frac{\frac{C_{u^2}p + C_{ud}(1-p)}{1+r}p + \frac{C_{ud}p + C_{d^2}(1-p)}{1+r}(1-p)}{1+r}$$

or

$$C = \frac{C_{u^2}p^2 + 2C_{ud}p(1-p) + C_{d^2}(1-p)^2}{(1+r)^2}.$$

This is the price of the call with two periods to expiration.

Exercise

Complete the three-step binomial tree and express C as a function of $C_{u^3}, C_{d^3}, C_{u^2d}, C_{ud^2}, p, (1-p), r$.

In general if we divide the time to expiration into n periods we get

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{n-j}}}{(1+r)^n}, \text{ where } C_{u^j d^{n-j}} = \max(u^j d^{n-j} S_0 - E, 0).$$

This expression can be simplified because the call is not always in the money at the end of the nth period. For the call to be in the money we want a minimum of k up movements of the stock. Therefore if we find k the summation will simply begin from $j = k$. The call is in the money as long as $u^k d^{n-k} S_0 - E > 0$ and solving for k we get $k = \frac{log(\frac{E}{d^n S_0})}{log(\frac{u}{d})}$. And the price of the call is equal to:

$$\begin{aligned}
C &= \frac{\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{n-j}}}{(1+r)^n} = \frac{\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} C_{u^j d^{n-j}}}{(1+r)^n} \\
C &= \frac{\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left(u^j d^{n-j} S_0 - E\right)}{(1+r)^n} \\
C &= S_0 \left[\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \frac{(pu)^j [(1-p)d]^{n-j}}{(1+r)^n}\right] - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}
\end{aligned}$$

Aside: If we let $p' = \frac{pu}{1+r}$ then $\frac{(pu)^j [(1-p)d]^{n-j}}{(1+r)^n} = p'(1-p')^{n-j}$ and the price of the call is equal to:

$$\begin{aligned}
C &= S_0 \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p'^j (1-p')^{n-j} - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
C &= S_0 P(X \geq k) - \frac{E}{(1+r)^n} P(Y \geq k), \text{ where } X \sim b(n, p'), \text{and } Y \sim b(n, p).
\end{aligned}$$

Numerical example
Using the binomial option pricing model find the price of a European call if $S_0 = \$30, E = \$29, \sigma = 0.30, r = 0.05$, with 73 days to expiration ($\frac{1}{5}$ of a year), and $n = 5$ periods (five-step binomial tree).

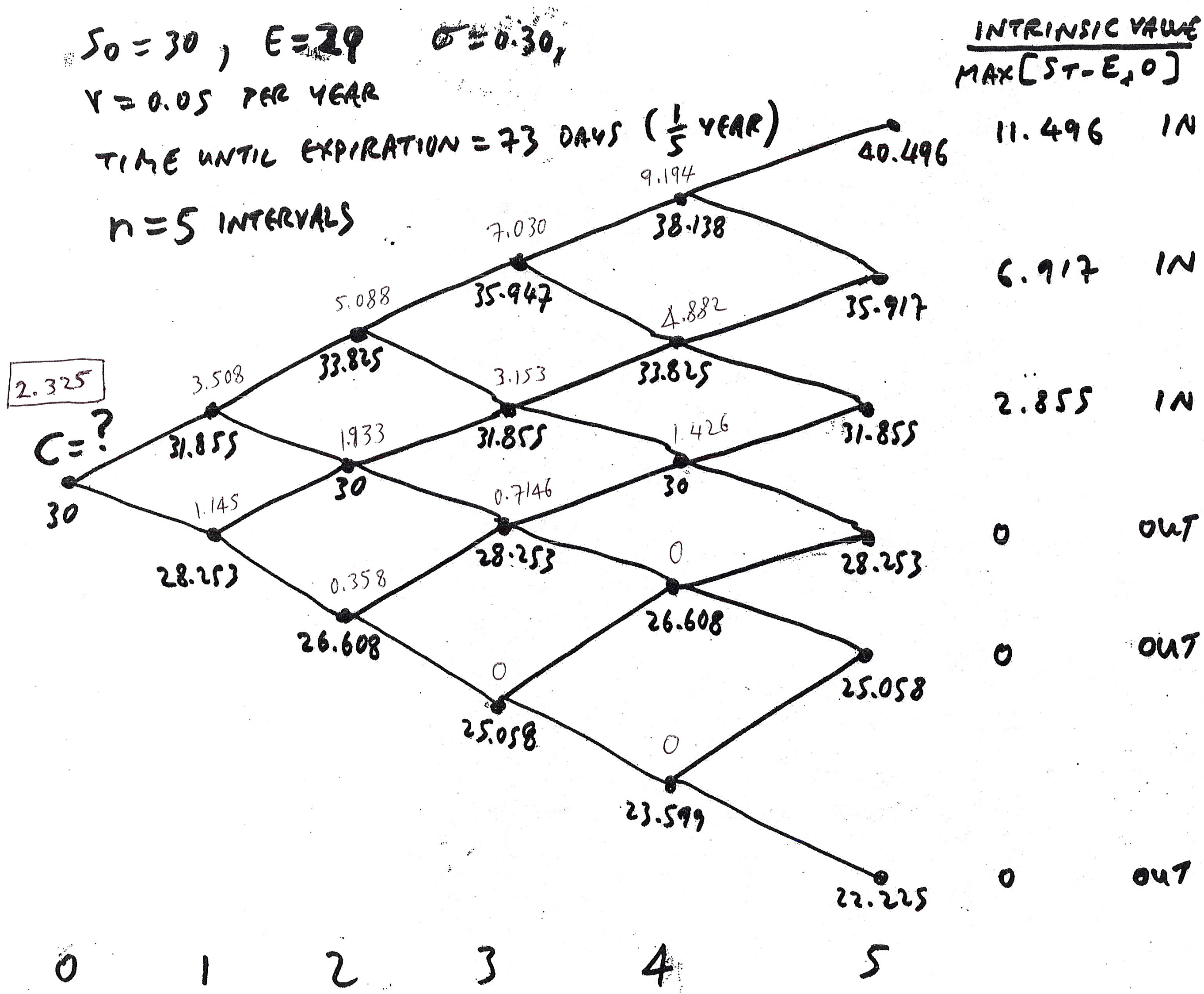
$$\begin{aligned}
u &= e^{\sigma\sqrt{\frac{t}{n}}} = exp(0.30\sqrt{\frac{0.2}{5}}) = 1.061837. \\
d &= e^{-\sigma\sqrt{\frac{t}{n}}} = \frac{1}{u} = 0.941764. \\
r_p &= (1.05)^{\frac{1}{25}} - 1 = 0.001954. \\
p &= \frac{1 + r_p - d}{u - d} = 0.50128. \\
p' &= \frac{pu}{1 + r_p} = 0.53124. \\
k &= \frac{log(\frac{E}{d^n S_0})}{log(\frac{u}{d})} = 2.22 \Rightarrow k = 3. \\
C &= 30P(X \geq 3) - \frac{29}{1 + 0.001954)^5} P(Y \geq 3) = 2.325.
\end{aligned}$$

Note: $X \sim b(5, 0.53124$ and $Y \sim b(5, 0.50128)$.

The `R` command:

```
30*pbinom(2,5,0.53124, lower.tail=FALSE) -
(29/(1+.001954)^5)*pbinom(2, 5, 0.50128, lower.tail=FALSE)
```

See next page for the complete five-step binomial tree.


S0 = 30 , E = 29 σ = 0.30,
r = 0.05 PER YEAR
TIME UNTIL EXPIRATION = 73 DAYS (1/5 YEAR)
n = 5 INTERVALS
INTRINSIC VALUE
MAX[ST - E, 0]
2.325
C = ?
30
3.508
31.855
1.145
28.253
5.088
33.825
1.933
30
0.358
26.608
7.030
35.947
3.153
31.855
0.7146
28.253
0
25.058
9.194
38.138
4.882
33.825
1.426
30
0
26.608
0
23.599
40.496
35.917
31.855
28.253
25.058
22.225
11.496 IN
6.917 IN
2.855 IN
0 OUT
0 OUT
0 OUT
0
1
2
3
4
5