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Statistics 19 **Instructor: Nicolas Christou**

Constructing the optimal portfolios - Constant correlation model
Short sales allowed

The calculation of optimal portfolios is simplified by using the constant correlatation model to rank securities based on the excess return to standard deviation ratio.

$$\text{Excess return to standard deviation} = \frac{\bar{R}_i - R_f}{\sigma_i}.$$

After stocks are ranked using the above ratio the optimum portfolio (point of tangency) consists of investing in all stocks: Those for which the excess return to beta is greater than the cut-off point C^* will be held long. Those for which the excess return to beta is smaller than the cut-off point C^* will be held short. This cut-off point is computed as follows:

$$C^* = \frac{\rho}{1-\rho+N\rho}\sum_{j=1}^{N}\frac{\bar{R}_j - R_f}{\sigma_j}$$

where

$\bar{R}_j$	Expected return on stock j.
R_f	Return on a riskless asset.
σ_j	Standard deviation of the returns of stock j.
ρ	The correlation coefficient - it is constant for all pairs of stocks.

To find the proportion of funds invested in each of these stocks we use:

$$z_i = \frac{1}{(1-\rho)\sigma_i}\left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right)$$

Therefore

$$x_i = \frac{z_i}{\sum_{i=1}^{N} z_i}.$$

where N is equal to the number of stocks consisting the optimum portfolio (here all stocks because short sales are allowed).
Below an example with 10 stocks is presented. The first table shows the expected return, standard deviation, and the excess return to standard deviation ratio assuming $R_f = 0.02$. The second table shows the procedure for the calculation of the cut-off point C^* and the optimum portfolio by assuming $\rho = 0.2$ for all pairs of stocks.

Stock i	$\bar{R}_i$	$\bar{R}_i - R_f$	σ_i	$\frac{\bar{R}_i - R_f}{\sigma_i}$
1	0.09	0.07	0.015	4.667
2	0.13	0.11	0.025	4.400
3	0.08	0.06	0.02	3.000
4	0.12	0.10	0.04	2.500
5	0.08	0.06	0.03	2.000
6	0.15	0.13	0.10	1.300
7	0.17	0.15	0.13	1.154
8	0.10	0.08	0.08	1.000
9	0.05	0.03	0.035	0.857
10	0.06	0.04	0.06	0.667

Using the above table we compute the entries in the next table. The last column contains the C^*.

Stock i	$\frac{\bar{R}_i - R_f}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	$\sum_{j=1}^{i} \frac{\bar{R}_j - R_f}{\sigma_j}$	C^*
1	4.667	0.2	4.667	
2	4.400	0.167	9.067	
3	3.000	0.143	12.067	
4	2.500	0.125	14.567	
5	2.000	0.111	16.567	
6	1.300	0.100	17.867	
7	1.154	0.091	19.021	
8	1.000	0.083	20.021	
9	0.857	0.077	20.878	
10	0.667	0.071	21.544	1.539

We find from the previous table that $C^* = 1.539$. Therefore in the optimum portfolio (point of tangency) the first 5 ranked stocks will be held long and the last 5 will be held short.
We first find the values of z_i's:

$$z_1 = \frac{1}{(1-\rho)\sigma_1}\left(\frac{\bar{R}_1 - R_f}{\sigma_1} - C^*\right) = \frac{1}{(1-0.2)0.015}(4.667 - 1.539) = 260.67.$$

Similarly, $z_2 = 143.05$, $z_3 = 91.31$, $z_4 = 30.03$, $z_5 = 19.21$, $z_6 = -2.99$, $z_7 = -3.70$, $z_8 = -8.42$, $z_9 = -24.36$, and $z_{10} = -18.17$. The sum of the z_i's is $\sum_{i=1}^{10} z_i = 486.63$. Therefore $x_1 = \frac{260.67}{486.63} = 0.535$, $x_2 = \frac{143.05}{486.63} = 0.294$, and similarly $x_3 = 0.188$, $x_4 = 0.062$, $x_5 = 0.039$, $x_6 = -0.006$, $x_7 = -0.007$, $x_8 = -0.017$, $x_9 = -0.050$, $x_{10} = -0.037$.

We conclude that the optimum portfolio consists of
53.5% of stock 1,
29.4% of stock 2,
18.8% of stock 3,
6.2% of stock 4,
3.9% of stock 5.
−0.6% of stock 6.
−0.7% of stock 7.
−1.7% of stock 8.
−5.0% of stock 9.
−3.7% of stock 10.