The residuals and their variance-covariance matrix

We have seen that the variance-covariance matrix of the residuals can be expressed as follows:

$$\text{cov}(\mathbf{e}) = \text{cov}(\mathbf{Y} - \hat{\mathbf{Y}}) = \text{cov}\left(\mathbf{Y} - \mathbf{HY}\right) = \text{cov}((\mathbf{I} - \mathbf{H})\mathbf{Y}) = (\mathbf{I} - \mathbf{H})\sigma^2\mathbf{I}(\mathbf{I} - \mathbf{H})' \Rightarrow$$

$$\text{cov}(\mathbf{e}) = \sigma^2(\mathbf{I} - \mathbf{H}).$$

Or if we expand this we get:

$$\text{cov}(\mathbf{e}) = \begin{pmatrix} \text{var}(e_1) & \text{cov}(e_1, e_2) & \text{cov}(e_1, e_3) & \cdots & \cdots & \text{cov}(e_1, e_n) \\ \text{cov}(e_2, e_1) & \text{var}(e_2) & \text{cov}(e_2, e_3) & \cdots & \cdots & \text{cov}(e_2, e_n) \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ \text{cov}(e_n, e_1) & \text{cov}(e_n, e_2) & \text{cov}(e_n, e_3) & \cdots & \cdots & \text{var}(e_n) \end{pmatrix} \Rightarrow$$

$$\text{cov}(\mathbf{e}) = \sigma^2 \begin{pmatrix} 1 - h_{11} & -h_{12} & -h_{13} & \cdots & \cdots & -h_{1n} \\ -h_{21} & 1 - h_{22} & -h_{23} & \cdots & \cdots & -h_{2n} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ -h_{n1} & -h_{n2} & -h_{n3} & \cdots & \cdots & 1 - h_{nn} \end{pmatrix}$$

Where, $1 - h_{ij}$ is the ij_{th} element of the matrix $\mathbf{I} - \mathbf{H}$. Therefore the variance of the i_{th} residual is $\text{var}(e_i) = \sigma^2(1 - h_{ii})$. Since the variance is always ≥ 0 we have $1 - h_{ii} \geq 0 \Rightarrow h_{ii} \leq 1$. If h_{ii} is close to 1 the variance of the i_{th} residual will be very small which means that the fitted line is forced to pass near the point that corresponds to this residual (small variance of a residual means that $\hat{y}_i$ is close to the observed y_i). In the extreme case when $h_{ii} = 1$ the fitted line will definitely pass through point i because $\text{var}(e_i) = 0$.