Moment generating functions

Definition:

\[ M_X(t) = Ee^{tX} \]

Therefore,

If \( X \) is discrete

\[ M_X(t) = \sum_x e^{tX}P(x) \]

If \( X \) is continuous

\[ M_X(t) = \int_x e^{tX} f(x)dx \]

Aside:

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

Similarly,

\[ e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots \]

Let \( X \) be a discrete random variable.

\[ M_X(t) = \sum_x e^{tX}P(x) = \sum_x \left[ 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots \right] P(x) \]

or

\[ M_X(t) = \sum_x P(x) + \frac{t}{1!} \sum_x xP(x) + \frac{t^2}{2!} \sum_x x^2P(x) + \frac{t^3}{3!} \sum_x x^3P(x) + \cdots \]

To find the \( k \)th moment simply evaluate the \( k \)th derivative of the \( M_X(t) \) at \( t = 0 \).

\[ EX^k = \left[ M_X(t) \right]_{t=0}^{k \text{th derivative}} \]

For example:

First moment:

\[ M_X(t)' = \sum_x xP(x) + \frac{2t}{2!} \sum_x x^2P(x) + \cdots \]

We see that \( M_X(0)' = \sum_x xP(x) = E(X) \).
Similarly, second moment

\[ M_X(t)'' = \sum_x x^2 P(x) + \frac{6t}{3!} \sum_x x^3 P(x) + \cdots \]

We see that \( M_X(0)'' = \sum_x x^2 P(x) = E(X^2) \).

Examples:
Find the moment generating function of \( X \sim b(n,p) \).
Find the moment generating function of \( X \sim \text{Poisson}(\lambda) \).
Find the moment generating function of \( X \sim \text{exp}(\lambda) \).
Find the moment generating function of \( Z \sim N(0,1) \).
Theorem:
Let $X, Y$ be independent random variables with moment generating functions $M_X(t), M_Y(t)$ respectively. Then, the moment generating function of the sum of these two random variables is equal to the product of the individual moment generating functions:

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

Proof:

Let $X, Y$ independent random variables.
Use this theorem to find the distribution of $X + Y$, where $X \sim b(n_1, p), Y \sim b(n_2, p)$.
Use this theorem to find the distribution of $X + Y$, where $X \sim \text{Poisson}(\lambda_1), Y \sim \text{Poisson}(\lambda_2)$.
Use this theorem to find the distribution of $X + Y$, where $X \sim N(\mu_1, \sigma_1), Y \sim N(\mu_2, \sigma_2)$. 
Properties of moment generating functions:
Let $X$ be a random variable with moment generating function $M_X(t) = Ee^{tX}$, and $a, b$ are constants

1. $M_{X+a}(t) = e^{at}M_X(t)$
2. $M_{bX}(t) = M_X(bt)$
3. $M_{\frac{X+a}{b}} = e^{\frac{at}{b}}M_X(\frac{t}{b})$

Proof:

Use these properties and the moment generating function of $Z \sim N(0, 1)$ to find the moment generating function of $X \sim N(\mu, \sigma)$
Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from $N(\mu, \sigma)$. Use moment generating functions to find the distribution of

a. $T = X_1 + X_2 + \ldots + X_n$.

b. $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$.
Distribution of the sample mean - Sampling from normal distribution

If we sample from normal distribution $N(\mu, \sigma)$ then $\bar{X}$ follows exactly the normal distribution with mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$ regardless of the sample size $n$. In the next figure we see the effect of the sample size on the shape of the distribution of $\bar{X}$. The first figure is the $N(5, 2)$ distribution. The second figure represents the distribution of $\bar{X}$ when $n = 4$. The third figure represents the distribution of $\bar{X}$ when $n = 16$. 

![Graphs of normal distributions with different sample sizes](image_url)