UCLA Department of Statistics
Edward R. Roybal Learning Center

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Stock market portfolios

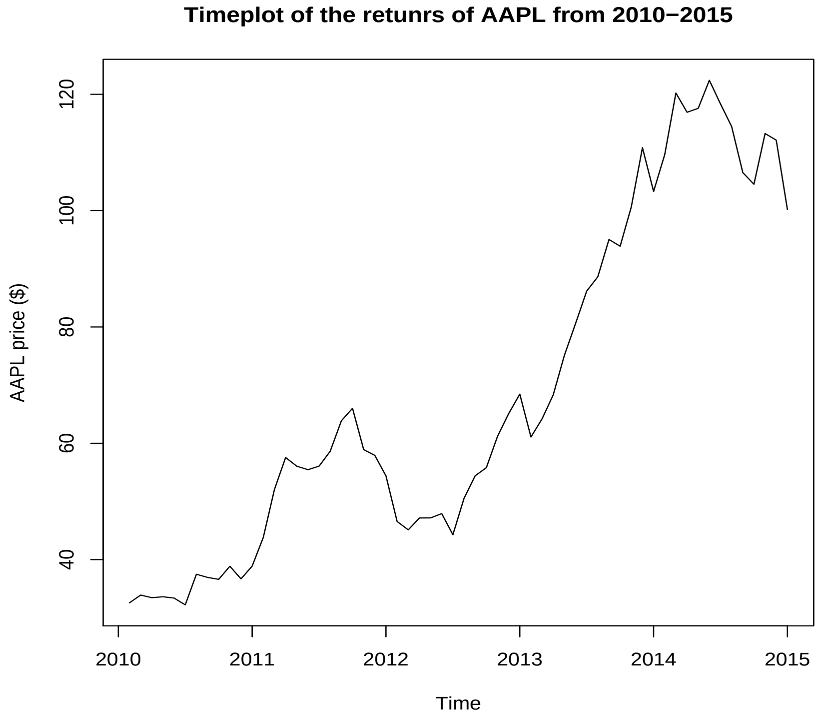
- At the end of each trading day a stock has an "adjusted close price".

- Adjusted close prices and other numbers are recorded for all stocks and stored for each trading day.

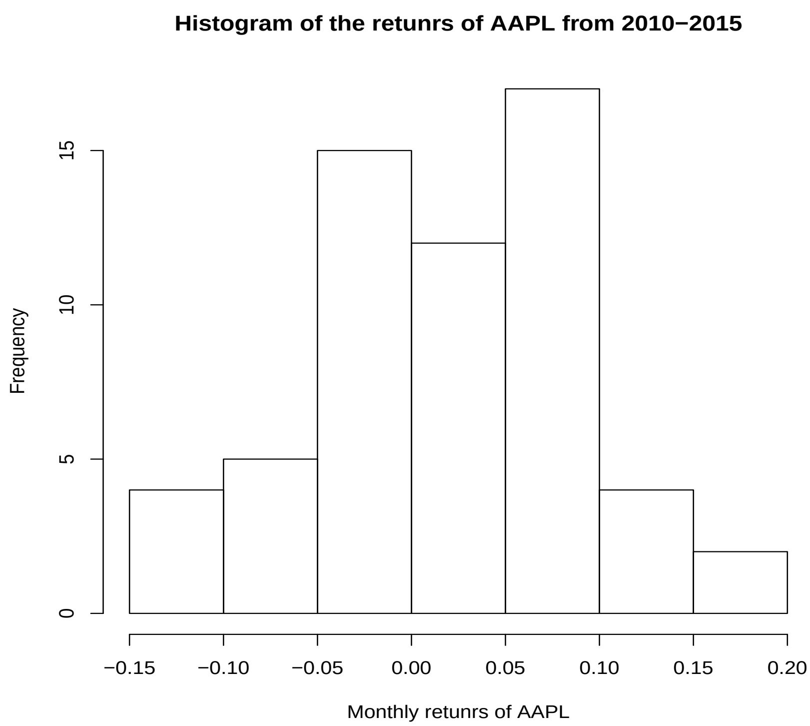
- We convert these adjusted close prices into "returns". How? Suppose a stock had an adjusted close price of \$50 at the end of January 2013 and at the end of February 2013 an adjusted close price of \$51. What is the return of the stock during this period?

- For our project we will collect historical data on the $S\&P500$ index and on three stocks: C, $AAPL$, and MCD from December 2012 to December 2015.

- We plot the adjusted close prices of each stock against time to see how they fluctuate (see example for $AAPL$ on the next page). This is called a timeplot. What do you observe?



- We can also construct the histogram of the returns of each stock (for example see below the histogram for $AAPL$). What do you observe?



- Measure of central tendency: Where do the returns tend to center?
  Mean: Simply add all the returns and divide by how many they are. Mathematically, this can be written as

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_n}{n} \quad \text{or } \bar{R} = \frac{\sum_{i=1}^{n} R_i}{n}.$$

- Measure of variation: A number that tells us how the returns vary around the mean (around the center). Mathematically, this is equal to

$$\sigma^2 = \frac{(R_1 - \bar{R})^2 + (R_2 - \bar{R})^2 + \cdots + (R_n - \bar{R})^2}{n} \quad \text{or } \sigma^2 = \frac{\sum_{i=1}^{n}(R_i - \bar{R})^2}{n}.$$

  This is called the variance. The square root of the variance is the standard deviation.

- For our example, the two stocks have the following means, variances, and standard deviations:

	$S\&P500$	C	$AAPL$	MCD
Mean	0.008447	0.005112	0.021579	0.011580
Variance	0.001157	0.007632	0.004853	0.001314
Standard deviation	0.034015	0.087361	0.069663	0.036249

- Very important: The returns of the two stocks are associated. This association is measured with the covariance. Mathematically, this is equal to:

$$\sigma_{AB} = \frac{(R_1 - \bar{R}_A)(R_1 - \bar{R}_B) + \cdots + (R_{60} - \bar{R}_A)(R_{60} - \bar{R}_B)}{n}$$

  or

$$\sigma_{AB} = \frac{\sum_{i=1}^{n}(R_{iA} - \bar{R}_A)(R_{iB} - \bar{R}_B)}{n}$$

- In our example, the variances and covariances can be presented in the so called variance covariance matrix:

```
         X.GSPC        C     AAPL      MCD
X.GSPC 0.001157 0.002252 0.001076 0.000568
C      0.002252 0.007632 0.001625 0.000521
AAPL   0.001076 0.001625 0.004853 0.000124
MCD    0.000568 0.000521 0.000124 0.001314
```

- Why do we need all these in the theory of stock market portfolio? The means, variances, and covariances are called the **inputs of the portfolio**.

- Suppose we would like to construct portfolios using C and $AAPL$. An investor can choose many combinations of the two stocks to invest his/her money. For example:

  1. 100% in C and 0% in $AAPL$.
  2. 90% in C and 10% in $AAPL$.
  3. 60% in C and 40% in $AAPL$.
  4. 20% in C and 80% in $AAPL$.
  5. 0% in C and 100% in $AAPL$.

  etc.

- Each one of these combinations have a mean return and variance.

- Let x_A be the fraction of investor's funds invested into stock A (C), and
  Let x_B be the fraction of investor's funds invested into stock B ($AAPL$).

- Mean return of a portfolio:

$$\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$$

- Variance (risk) of a portfolio:

$$\sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \ \sigma_{AB}$$

- Let's do a numerical example. Find the mean and variance of a portfolio that consists of 20% C and 80% $AAPL$ (the fractions are $x_A = 0.20, x_B = 0.80$).

$$\bar{R}_p = 0.20(0.005112) + 0.80(0.021579) = 0.018286.$$

$$\sigma_p^2 = 0.20^2(0.007632) + 0.80^2(0.004853) + 2(0.2)(0.8)0.001625 = 0.003931.$$

  We can take the square root of the variance to find the standard deviation of the portfolio $\sigma_p = \sqrt{0.003931} = 0.062697$.

- Similarly we can use many other combinations of x_A, x_B. Try the following: Find the mean return of the portfolio and the standard deviation of the portfolio for each one of the following combinations of x_A, x_B.

x_A	x_B	$\bar{R}_p$	σ_p
0.00	1.00		
0.05	0.95		
0.10	0.90		
0.15	0.85		
0.20	0.80	0.018286	0.062697
0.25	0.75		
0.30	0.70		
0.35	0.65		
0.40	0.60		
0.45	0.55		
0.50	0.50		
0.55	0.45		
0.60	0.40		
0.65	0.35		
0.70	0.30		
0.75	0.25		
0.80	0.20		
0.85	0.15		
0.90	0.10		
0.95	0.05		
1.00	0.00		

Hint:
$\bar{R}_A = 0.005112$
$\bar{R}_B = 0.021579$
$\sigma_A^2 = 0.007632$
$\sigma_B^2 = 0.004853$
$\sigma_{AB} = 0.001625$

- Now let's plot the mean return of the portfolio ($\bar{R}_p$) against the standard deviation (risk) of the portfolio (σ_p).

Here are the results.

```
     xa   xb       sd_p      mean_p
1  0.00 1.00 0.06966315 0.021579070
2  0.05 0.95 0.06747792 0.020755739
3  0.10 0.90 0.06557253 0.019932408
4  0.15 0.85 0.06397199 0.019109076
5  0.20 0.80 0.06269963 0.018285745
6  0.25 0.75 0.06177575 0.017462414
7  0.30 0.70 0.06121612 0.016639083
8  0.35 0.65 0.06103077 0.015815751
9  0.40 0.60 0.06122309 0.014992420
10 0.45 0.55 0.06178956 0.014169089
11 0.50 0.50 0.06272004 0.013345758
12 0.55 0.45 0.06399865 0.012522426
13 0.60 0.40 0.06560505 0.011699095
14 0.65 0.35 0.06751584 0.010875764
15 0.70 0.30 0.06970600 0.010052433
16 0.75 0.25 0.07215008 0.009229102
17 0.80 0.20 0.07482321 0.008405770
18 0.85 0.15 0.07770175 0.007582439
19 0.90 0.10 0.08076375 0.006759108
20 0.95 0.05 0.08398913 0.005935777
21 1.00 0.00 0.08735981 0.005112445
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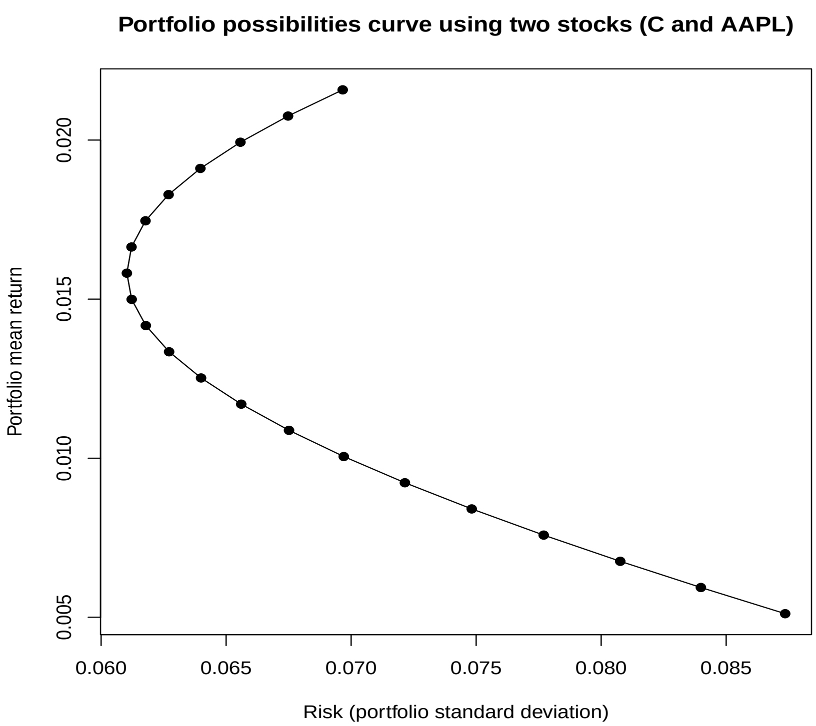

Extension:
What if we want to find the portfolio with the minimum risk?
This is the mathematical formulation of the problem. We first want to minimize the variance of the portfolio. This will be:

$$\begin{aligned} \text{Minimize} \;\; \sigma_p^2 = x_A^2\sigma_A^2 + x_B\sigma_B^2 + 2x_Ax_B \; \sigma_{AB} \\ \text{subject to} \;\; x_A + x_B = 1 \end{aligned}$$

Therefore our goal is to find x_A and x_B, the fractions of the available funds that will be invested in each stock. Substituting $x_B = 1 - x_A$ into the equation of the variance we get

$$\sigma_p^2 = x_A^2\sigma_A^2 + (1 - x_A)\sigma_B^2 + 2x_A(1 - x_A) \; \sigma_{AB}$$

To minimize the above expression we take the derivative with respect to x_A, set it equal to zero and solve for x_A. The result is:

$$x_A = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$$

and therefore

$$x_B = 1 - x_A = \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$$

Example with 3 stocks: