University of California, Los Angeles
Department of Statistics

Statistics 100A

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Exam 1
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Name: SOLUTIONS

Problem 1 (15 points)
Three cards are identical in form except that both sides of the first one are colored green, both sides of the second card are colored blue, and one side of the third one is colored green and the other side blue. The three cards are mixed up in a hat, and one card is randomly selected and placed down on the ground. If the upper side of the chosen card is colored green, what is the probability that the other side is colored blue?

$$P[BG \mid G] = \frac{P[BG \cap G]}{P(G)}$$

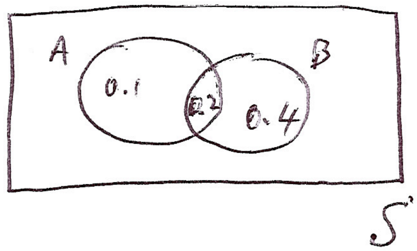
$$= \frac{P(BG \cap G)}{P(G \cap GG) + P(G \cap BG) + P(G \cap BB)}$$

$$= \frac{P(G \mid BG) \cdot P(BG)}{P(G \mid GG) \cdot P(GG) + P(G \mid BG) \cdot P(BG) + P(G \mid BB) P(BB)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{2}\frac{1}{3} + 0\frac{1}{3}} = \frac{1}{3}.$$

Problem 2 **(35 points)**
Answer the following questions:

a. Suppose $P(A) = 0.3$, $P(B) = 0.6$, and $P(A \cap B) = 0.2$. Find $P(A'|B)$.



$$P(A'|B) = \frac{P(A' \cap B)}{P(B)} = \frac{0.4}{0.6} = \frac{2}{3}.$$

b. You roll two dice repeatedly until you observe the sum of 10 or the sum of 7. What is the probability that the sum of 10 is obtained before the sum of 7?

$$P(10 \mid 10 \text{ OR } 7) = \frac{P(10 \cap 10 \text{ OR } 7)}{P(10 \text{ OR } 7)} = \frac{P(10)}{P(10 \text{ OR } 7)} =$$

$$= \frac{3/36}{3/36 + 6/36} = \frac{1}{3}.$$

c. The probability mass function of a random variable X is given by $p(x) = \frac{c\lambda^x}{x!}$, with $x = 0, 1, 2, \ldots$. Find the constant c.

$$\sum_{x=0}^{\infty} c \frac{\lambda^x}{x!} = 1 \Rightarrow c \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1$$

$$\Rightarrow c\left[1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right) = 1$$

$$\Rightarrow c e^{\lambda} = 1 \Rightarrow \underline{c = e^{-\lambda}}.$$

d. Refer to part (c). Find $P(X > 2)$.

$$P(X > 2) = 1 - P(X \le 2)$$

$$= 1 - \left[P(X=0) + P(X=1) + P(X=2)\right]$$

$$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right)$$

$$= 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda}\right].$$

Problem 3 **(25 points)**
Answer the following questions:

a. Show that the determination of negative binomial probabilities can be simplified by making use of the identity

$$P(X = x) = \frac{r}{x} P(Y = r)$$

where X follows the negative binomial distribution and $Y \sim b(x, p)$. As a reminder, in the negative binomial distribution, X represents the number of trials needed until r successes occur (each trial has probability of success p).

$$\frac{r}{x}\binom{x}{r} p^r (1-p)^{x-r} = \frac{r}{x} \frac{x!}{(x-r)!\,r!} p^r (1-p)^{x-r}$$

$$= \frac{(x-1)!}{(x-r)!\,(r-1)!} p^r (1-p)^{x-r} = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$= P(X = x).$$

b. A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first, find the probability that the final roll is made by A.

$$P(A \text{ WINS}) = P\left(\substack{A \text{ WINS} \\ \text{ON FIRST}}\right) + \cdots$$

$$= \underset{A}{\frac{4}{36}} + \underset{A' \; B' \; A}{\frac{32}{36}\,\frac{31}{36}\,\frac{4}{36}} + \underset{A'B'A'B' \; A}{\left[\frac{32}{36} \cdot \frac{31}{36}\right]^2 \frac{4}{36}} + \cdots = \frac{4/36}{1 - \frac{32}{36} \cdot \frac{31}{36}}$$

c. Suppose the number X of internet users that visit a particular website follow the Poisson distribution with parameter $\lambda = 3$ per minute. Compute $P(X > 2 | X > 1)$.

$$P(X > 2 / X > 1) = \frac{P(X > 2 \cap X > 1)}{P(X > 1)}$$

$$= \frac{P(X > 2)}{P(X > 1)} = \frac{1 - P(X \leq 2)}{1 - P(X \leq 1)}$$

$$= \frac{1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!}\right]}{1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!}\right]} = \frac{1 - 13e^{-3}}{1 - 4e^{-3}}$$

Problem 4 **(25 points)**
Answer the following questions:

a. Customers arrive independently at a cashier. The probability that a customer pays with cash is 40%. Find the probability that the 12_{th} customer is the 8_{th} that pays with cash.

$p = 0.4$ NEGATIVE BINOMIAL

$$P(X=12) = \binom{12-1}{8-1} 0.4^8\, 0.6^4 = \binom{11}{7} 0.4^8\, 0.6^4$$

$$= 0.02803.$$

b. Cards are selected with replacement until the first ace is found. Given that the first ace is found on or after the 5_{th} trial, what is the probability that the first ace is found after the 8_{th} trial?

$$P(X>8 \mid X \geq 5) = P(X>8 \mid X>4)$$

$= P(X>4)$ MEMORYLESS PROPERTY

$$= \left(\frac{48}{52}\right)^4.$$

c. Ten cards are selected with replacement. What is the probability that 4 of them are clubs?

BINOMIAL

$$P(X=4) = \binom{10}{4}\left(\frac{13}{52}\right)^4\left(\frac{39}{52}\right)^6 = 0.146.$$