University of California, Los Angeles
Department of Statistics

Statistics 100A — Instructor: Nicolas Christou

Exam 2
16 November 2010

Name: SOLUTIONS

Problem 1 (25 points)
Answer the following questions:

a. Many people cancel their reservations at hotels at the last minute. Therefore, most hotels overbook when possible. That is, they make more reservations that they have rooms. A particular hotel has 90 rooms, and, on the average, 20% of the people with reservations cancel them. Suppose that for a particular night the hotel has 100 reservations. Approximate the probability that more people will show up with reservations than they have rooms for?

$$\mu = np = 100(0.8) = 80$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{100(0.8)(0.2)} = 4$$

$$P(X > 90) = P\left(Z > \frac{90.5 - 80}{4}\right) = P(Z > 2.63)$$

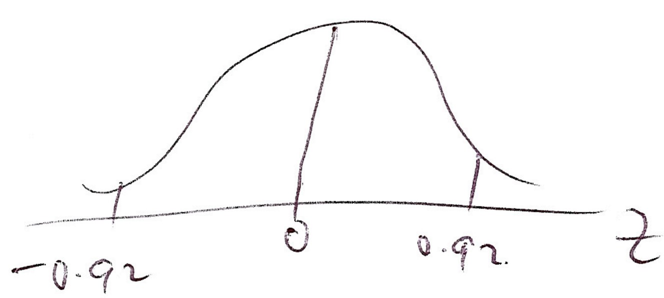
$$= 1 - 0.9957 = \underline{\underline{0.0043.}}$$

b. Suppose that accidents on a particular road occur as a Poisson process at a rate of two per month. Approximate the probability that in 18 months there will be more than 30 accidents?

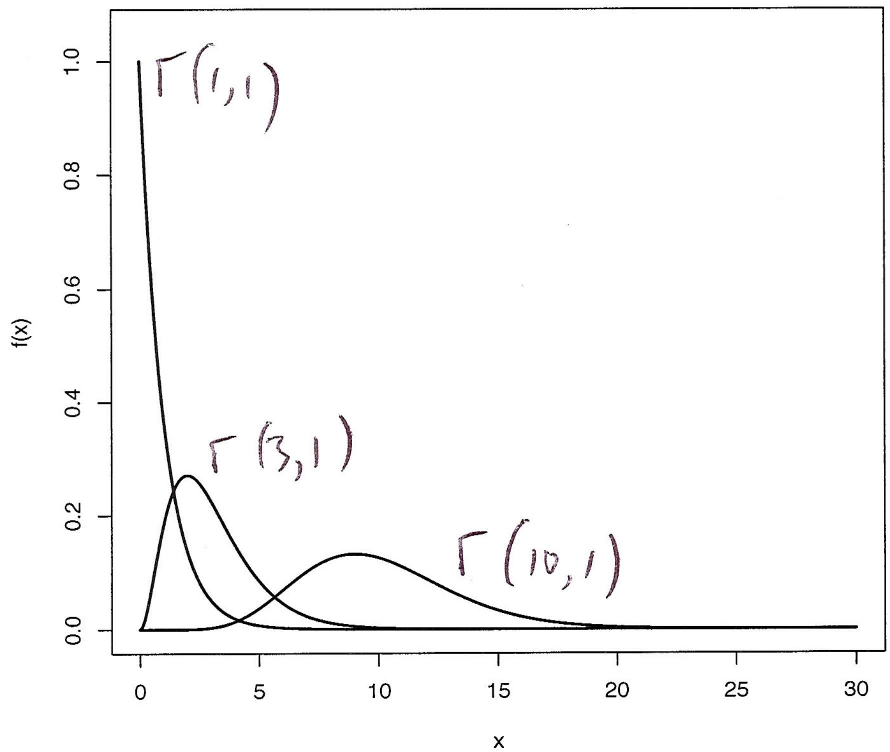
$$\lambda_{18} = 36 \qquad \mu = 36, \quad \sigma = \sqrt{36} = 6$$

$$P(X > 30) = P\left(Z > \frac{30.5 - 36}{\sqrt{36}}\right) = P(Z > -0.92)$$

$$= P(Z < 0.92) = 0.8212.$$



Problem 2 - Question (a):



Problem 2 (25 points)
Answer the following questions:

a. The graph on the previous page shows three gamma densities, $\Gamma(3,1), \Gamma(10,1)$, and $\Gamma(1,1)$. Please identify which one is which.

b. Suppose that X follows the uniform distribution on (0, n). Find the mean and variance of X.

$$E(X) = \frac{n}{2}, \qquad \text{var}(X) = \frac{n^2}{12}$$

c. Refer to question (b). Show that

$$Y = \frac{X - \mu_X}{\sigma_X}$$

$$f(x) = \frac{1}{n}$$

follows also the uniform distribution on the interval $(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$ with pdf $f(y) = \frac{1}{\sqrt{12}}$.

$$F_y(y) = P(Y \le y) = P\left(\frac{(X - \frac{n}{2})\sqrt{12}}{n} \le y\right] = P\left[X \le \frac{n}{2} + \frac{ny}{\sqrt{12}}\right]$$

$$\Rightarrow F_y(y) = F_X\left[\frac{n}{2} + \frac{ny}{\sqrt{12}}\right] \Rightarrow f_Y(y) = \frac{n}{\sqrt{12}} f_X\left(\frac{n}{2} + \frac{ny}{\sqrt{12}}\right)$$

$$= \frac{n}{\sqrt{12}} \cdot \frac{1}{n} \Rightarrow \underline{f_Y(y) = \frac{1}{\sqrt{12}}}$$

d. Since the standard normal density function integrates to 1 we should have:

$$1 = 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

By using $u = \frac{1}{2}z^2$ show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$du = z\,dz$$

$$z^2 = 2u \Rightarrow z = (2u)^{1/2}$$

$$1 = 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u} (2u)^{-1/2} du \Rightarrow \frac{\sqrt{2\pi}}{2 \cdot 2^{-1/2}} = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

$$\Rightarrow \underline{\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right)}$$

Problem 3 **(25 points)**
Answer the following questions:

a. Let $X \sim \Gamma(\alpha, \beta)$. Find EX^{-1}. (The gamma distribution is one of the few distributions for which EX^{-1} can be computed easily).

FROM CLASS NOTES : $EX^{k} = \dfrac{\Gamma(\alpha+k)\beta^{k}}{\Gamma(\alpha)}$

$$EX^{-1} = \frac{\Gamma(\alpha-1)\beta^{-1}}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)}\frac{1}{\beta} \Rightarrow \boxed{EX^{-1} = \frac{1}{(\alpha-1)\beta}}$$

b. Let $X \sim \Gamma(\alpha, \beta)$. Show that $Y = \frac{X}{\beta}$ follows $\Gamma(\alpha, 1)$.

$$F_Y(y) = P(Y \le y) = P\left[\frac{X}{\beta} \le y\right] = P(X \le \beta y)$$

$$\Rightarrow F_Y(y) = F_X(\beta y) \Rightarrow f_Y(y) = \beta \frac{(\beta y)^{\alpha-1} e^{-\frac{\beta y}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} = \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}$$

$$\Rightarrow f_Y(y) = \frac{y^{\alpha-1}e^{-\frac{y}{1}}}{\Gamma(\alpha)1^{\alpha}} \Rightarrow Y \sim \Gamma(\alpha, 1)$$

c. Suppose people move into a town as a Poisson process with parameter $\lambda = 5$ per week. Use gamma to write the expression that computes the probability that the 64th person will move to this town in less than 15 weeks.

0 ——— 15 WEEKS

$$f(t) = \frac{t^{63}e^{-5t}}{\Gamma(64)\left(\frac{1}{5}\right)^{64}} = \frac{5^{64}t^{63}e^{-5t}}{\Gamma(64)}$$

$T \sim \Gamma(64, \frac{1}{5})$

$$P(T \le 15) = \int_0^{15} \frac{5^{64}t^{63}e^{-5t}}{\Gamma(64)}\,dt$$

d. Approximate the probability of part (c) using the normal distribution.

$EX = \alpha\beta = 64\frac{1}{5} = 12.8$

$VAR(X) = \alpha\beta^2 = 64\frac{1}{25} = 2.56$

$SD(X) = \sqrt{2.56} = 1.6$

$$P(X \le 15) = P\left(Z > \frac{15-12.8}{1.6}\right)$$

$$= P(Z \le 1.38) = \underline{0.9162}$$

~~[illegible]~~

e. Use Poisson to write the expression that computes the exact probability of question (c).

NEED AT LEAST 64 ARRIVALS IN 15 WEEKS

$\lambda_{15} = 15 \times 5 = 75$

$$P(X \ge 64) = \sum_{x=64}^{\infty} \frac{75^x e^{-75}}{x!} = 1 - \sum_{x=0}^{63} \frac{75^x e^{-75}}{x!}$$

APPROX. $P(X \ge 64) = P\left(Z > \frac{63.5-75}{\sqrt{75}}\right) = P(Z > -1.33) = P(Z < 1.33)$

$= \underline{0.9082}$.

Problem 4 **(25 points)**
The pdf of a continuous random variable X is $f(x) = 2x$ for $0 \le x \le 1$.

a. Find the cdf of X.

$$F(x) = P(X \le x) = \int_0^x 2u\,du = u^2 \Big|_0^x$$

$$\Rightarrow \boxed{F(x) = x^2}$$

b. Find the 81st percentile of X.

$$P(X \le c) = 0.81$$

$$F(c) = 0.81 \Rightarrow c^2 = 0.81 \Rightarrow \boxed{c = 0.9}$$

c. Find the mean and variance of X.

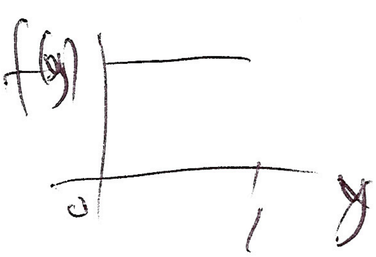
$$EX = \int_0^1 x\,2x\,dx = \int_0^1 2x^2\,dx = 2\frac{x^3}{3}\Big|_0^1$$

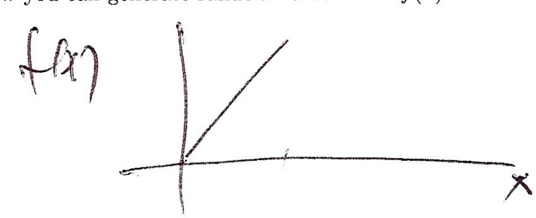
$$\Rightarrow \boxed{EX = \frac{2}{3}}$$

$$\mathrm{VAR}(X) = EX^2 - (EX)^2 = \int_0^1 x^2\,2x\,dx - \left(\frac{2}{3}\right)^2$$

$$= 2\frac{x^4}{4}\Big|_0^1 - \frac{4}{9} = \frac{1}{2} - \frac{4}{9} \Rightarrow \boxed{\mathrm{VAR}(X) = \frac{1}{18}}$$

d. Let $y_1, y_2, \ldots, y_n$ be the values of a random sample taken from the uniform (0,1) distribution. Use the inverse transformation method to explain how you can generate random values from $f(x) = 2x$.





$Y_1, \ldots, Y_n$

$X_1, \ldots, X_n$

$0 \le Y \le 1$

$0 \le F(x) \le 1$

$$Y = X^2 \Rightarrow \boxed{X = \sqrt{Y}}$$