University of California, Los Angeles
Department of Statistics

Statistics 100A — Instructor: Nicolas Christou

Exam 2
13 May 2011

Name: SOLUTIONS

$$f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}$$

Problem 1 (25 points)
Answer the following questions:

a. Let $X \sim \Gamma(\alpha, \beta)$. Show that $Y = cX$ follows $\Gamma(\alpha, c\beta)$.

$$F_Y(y) = P(Y \le y) = P(cX \le y) = P\left(X \le \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$

$$\Rightarrow f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c}\frac{y^{\alpha-1} e^{-\frac{y}{c\beta}}}{c^{\alpha-1}\beta^{\alpha}\Gamma(\alpha)} = \frac{y^{\alpha-1} e^{-\frac{y}{c\beta}}}{\Gamma(\alpha)(c\beta)^{\alpha}} \Rightarrow Y \sim \Gamma(\alpha, c\beta)$$

b. Let $X \sim N(\mu, \sigma)$. Find the distribution of $Y = e^X$.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = F_X(\ln y)$$

$$\Rightarrow f_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y}\frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{\ln y-\mu}{\sigma}\right)^2}$$

c. Suppose the radius of a circle X is a random variable that follows the exponential distribution with parameter λ. Find the distribution of the area of the circle: $Y = \pi X^2$.

$$f(x) = \lambda e^{-\lambda x}$$

$$F_Y(y) = P(Y \le y) = P(\pi X^2 \le y) = P\left(X^2 \le \frac{y}{\pi}\right)$$

$$= P\left(-\sqrt{\frac{y}{\pi}} \le X \le \sqrt{\frac{y}{\pi}}\right) = P\left(X \le \sqrt{\frac{y}{\pi}}\right) - \cancel{P\left(X \le -\sqrt{\frac{y}{\pi}}\right)} \to 0$$

$$= F_X\left(\sqrt{\frac{y}{\pi}}\right) \Rightarrow f_Y(y) = \frac{1}{2\sqrt{\pi y}} f_X\left(\sqrt{\frac{y}{\pi}}\right) \Rightarrow$$

$$f_X(x) = \frac{1}{2\sqrt{\pi y}} \lambda e^{-\lambda\sqrt{\frac{y}{\pi}}}$$

Problem 2 (25 points)
In California earthquakes of magnitude 1-2 in the Richter scale are recorded at the rate of 8 per hour according to a Poisson distribution. Answer the following questions:

a. What is the probability that more than 12 earthquakes (of magnitude 1-2 in the Richter scale) will be recorded in the next hour. Please write the expression that computes the exact probability (no computations).

$$P(X>12) = \sum_{x=13}^{\infty} \frac{8^x e^{-8}}{x!} = 1 - \sum_{x=0}^{12} \frac{8^x e^{-\lambda}}{x!}$$

b. Approximate the probability of part (a) using the normal distribution.

$$P(X>12) = P\left(Z > \frac{12.5-8}{\sqrt{8}}\right) = P(Z>1.59)$$

$$= 1-0.9441 = 0.0559$$

c. What is the distribution of the time that you have to wait until the 30th earthquake (of magnitude 1-2 in the Richter scale) is recorded. Please write the complete density of the distribution.

$$T \sim \Gamma\left(30, \frac{1}{8}\right) \qquad f(t) = \frac{8^{30} t^{30-1} e^{-8t}}{\Gamma(30)} = \frac{8^{30} t^{29} e^{-8t}}{\Gamma(30)}$$

d. Using the distribution of part (c) write the expression that computes the probability that the 30th earthquake (of magnitude 1-2 in the Richter scale) will be recorded in less than 4 hours from now.

$$P(T<4) = \int_0^4 \frac{8^{30} t^{29} e^{-8t}}{\Gamma(30)}\, dt$$

e. Approximate the probability of part (d).

$$E(T) = \alpha\beta = 30 \cdot \frac{1}{8} = 3.75$$

$$VAR(T) = \alpha\beta^2 = 30 \cdot \frac{1}{64} = 0.3 \Rightarrow SD(T) = 0.685$$

$$P(T<4) = P\left(Z < \frac{4-3.75}{0.685}\right) = P(Z<0.37) = 0.6443$$

f. Write the expression that computes the same probability of part (d) using the Poisson distribution (no computations).

$$P(T<4) = P(X \geq 30) = \sum_{x=30}^{\infty} \frac{32^x e^{-32}}{x!}$$

Problem 3 **(25 points)**
Answer the following questions:

a. Using your class notes and the one-page handout on the beta distribution show that the variance of beta distribution is

$$var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

FROM NOTES: $EX^k = \frac{B(\alpha+k, \beta)}{B(\alpha,\beta)}$

$EX = \frac{\alpha}{\alpha+\beta}$

$EX^2 = \frac{B(\alpha+2, \beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$

$= \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$ ✓

$var(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

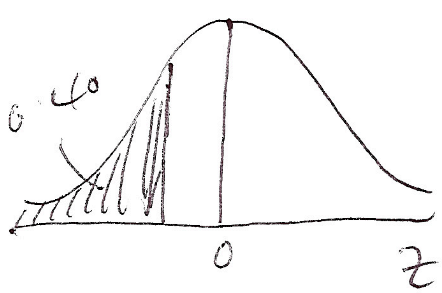
b. Scores on a certain standardized test, IQ scores, follow the normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 13$. An individual is selected at random. What is the probability that his score satisfies $120 < X < 130$?

$P(120 < X < 130) = P\left(\frac{120-100}{13} < Z < \frac{130-100}{13}\right)$

$= P(1.54 < Z < 2.31) = P(Z < 2.31) - P(Z < 1.54)$

$= 0.9896 - 0.9382 = \underline{0.0514}$

c. Refer to part (b). Find the 40th percentile of the distribution of these scores.



$-0.255 = \frac{c-100}{13}$

$c = 100 - (0.255)13$

$\underline{c = 96.685}$

Problem 4 (25 points)

Answer the following questions:

a. It is said that a random variable X follows the Pareto distribution with parameters x_0 and α with $x_0 > 0, \alpha > 0$, if X has the following probability density function:

$$f(x) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, \quad \text{for } x \geq x_0.$$

Show that $Y = ln(\frac{x}{x_0})$ follows the exponential distribution with parameter α. Note: x_0 is a constant.

$$F_Y(y) = P(Y \le y) = P[\ln \tfrac{X}{x_0} \le y] = P(\tfrac{X}{x_0} \le e^y) = P(X \le x_0 e^y)$$

$$F_Y(y) = F_X(x_0 e^y) \Rightarrow f_Y(y) = x_0 f_X(x_0 e^y)$$

$$= \frac{x_0 e^y \alpha x_0^\alpha}{x_0^{\alpha+1} e^{y(\alpha+1)}} = \frac{x_0^{\alpha+1}}{x_0^{\alpha+1}} \alpha e^y e^{-\alpha y - 1} \Rightarrow f_Y(y) = \alpha e^{-\alpha y}$$

$$\Rightarrow Y \sim \exp(\alpha).$$

b. Let $X \sim \Gamma(\alpha, \beta)$, with $\alpha > 2, \beta > 0$. Show that the variance of $\frac{1}{X}$ is $\frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}$. *Hint:* See your class notes for EX^k.

$$EX^k = \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \Rightarrow EX^{-1} = \frac{\beta^{-1}\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{\beta^{-1}\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} = \frac{1}{\beta(\alpha-1)}$$

$$EX^{-2} = \frac{\beta^{-2}\Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{\beta^{-2}\Gamma(\alpha-2)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}$$

$$\text{var}(X^{-1}) = EX^{-2} - (EX^{-1})^2 = \frac{1}{\beta^2(\alpha-1)(\alpha-2)} - \frac{1}{\beta^2(\alpha-1)^2}$$

$$= \frac{(\alpha-1)-(\alpha-2)}{\beta^2(\alpha-1)^2(\alpha-2)} = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}.$$

c. Let U be a uniform random variable on [0,1], and let $V = \frac{1}{U}$. Find the probability density function of V. For what values of V is this density valid? On the previous page please draw the density of U and the density of V on two separate graphs.

$$F_V(v) = P(V \le v) = P(\tfrac{1}{U} \le v) = P(U > \tfrac{1}{v}) \qquad v > 1$$

$$= 1 - P(U \le \tfrac{1}{v}) \Rightarrow F_V(v) = 1 - F_U(U \le \tfrac{1}{v})$$

$$= 1 - \tfrac{1}{v}$$

$$\Rightarrow f_V(v) = \frac{1}{v^2} \cancel{f_U(\tfrac{1}{v})}$$

$$\Rightarrow \boxed{f_V(v) = \frac{1}{v^2}}$$