

University of California, Los Angeles
Department of Statistics

Statistics 100A

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Homework 5

EXERCISES 1

Find the distribution of the random variable X for each of the following moment-generating functions:

a. $M_X(t) = \left[\frac{1}{3}e^t + \frac{2}{3}\right]^5$.

b. $M_X(t) = \frac{e^t}{2-e^t}$.

c. $M_X(t) = e^{2(e^t-1)}$.

EXERCISES 2

Let $M_X(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$ be the moment-generating function of a random variable X .

a. Find $E(X)$.

b. Find $\text{var}(X)$.

c. Find the distribution of X .

EXERCISES 3

Let X follow the Poisson probability distribution with parameter λ . Its moment-generating function is $M_X(t) = e^{\lambda(e^t-1)}$.

a. Show that the moment-generating function of $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ is given by:

$$M_Z(t) = e^{-\sqrt{\lambda}t} e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}.$$

b. Use the series expansion of

$$e^{\frac{t}{\sqrt{\lambda}}} = 1 + \frac{\frac{t}{\sqrt{\lambda}}}{1!} + \frac{\left(\frac{t}{\sqrt{\lambda}}\right)^2}{2!} + \frac{\left(\frac{t}{\sqrt{\lambda}}\right)^3}{3!} + \dots$$

to show that

$$\lim_{\lambda \rightarrow \infty} M_Z(t) = e^{\frac{1}{2}t^2}.$$

In other words, as $\lambda \rightarrow \infty$, the ratio $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ converges to the standard normal distribution.

EXERCISES 4

Use the result of part (b) of the previous exercise:

In the interest of pollution control an experimenter wants to count the number of bacteria per small volume of water. Let X denote the bacteria count per cubic centimeter of water, and assume that X follows the Poisson distribution with parameter $\lambda = 100$. If the allowable pollution in a water supply is a count of 110 bacteria per cubic centimeter, approximate the probability that X will be at most 110.

EXERCISES 5

Let X_1, X_2, \dots, X_n be i.i.d. random sample from $N(\mu, \sigma)$. Using moment generating functions show that the sum of these n observations $T = \sum_{i=1}^n X_i$ also follows the normal distribution. What is the mean and standard deviation of T ?

EXERCISES 6

Suppose that X_1, \dots, X_m and Y_1, \dots, Y_n are two samples, with $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$. The difference between the sample means, $\bar{X} - \bar{Y}$, is then a linear combination of $m + n$ normal random variables.

- a. Find $E(\bar{X} - \bar{Y})$.
- b. Find $Var(\bar{X} - \bar{Y})$.
- c. Use moment generating functions to show that the distribution of $\bar{X} - \bar{Y}$ is normal with mean and variance equal to your answers in (a) and (b).
- d. Suppose $\sigma_1^2 = 2, \sigma_2^2 = 2.5$, and $m = n$. Find the sample sizes so that $\bar{X} - \bar{Y}$ will be within one unit of $\mu_1 - \mu_2$ with probability 0.95.

EXERCISE 7

If X follows the geometric distribution with probability of success p , show that the moment-generating function for X is

$$M(t) = \frac{pe^t}{1 - qe^t}$$

where $q = 1 - p$.

EXERCISE 8

Differentiate the moment-generating function of a geometric random variable (see previous exercise) to find $E(X)$, and $E(X^2)$. Then find $Var(X)$.