

University of California, Los Angeles
Department of Statistics

Statistics 100A

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Homework 5

EXERCISE 1

A coin is tossed 3 times independently. One of the variables of interest is the number of tails X . Let Y denote the amount of money won on a side bet in the following manner:

If the first tail occurs on the first toss, you win \$1.

If the first tail occurs on the second toss, you win \$2.

If the first tail occurs on the third toss, you win \$3.

If no tails appear you lose \$1.

Construct the joint probability distribution of X and Y . In other words complete the following table where the entries are the probabilities for each pair of values of the variables X and Y .

	X			
Y	0	1	2	3
-1	?	?	?	?
1	?	?	?	?
2	?	?	?	?
3	?	?	?	?

EXERCISE 2

A die is rolled and the number observed X is recorded. Then a coin is tossed number of times equal to the value of X . For example if $X = 2$ then the coin is tossed twice, etc. Let Y be the number of heads observed. Note: Assume that the die and the coin are fair.

- Construct the joint probability distribution of X and Y .
- Find the conditional expected value of Y given $X = 5$.
- Find the conditional variance of Y given $X = 5$.

EXERCISE 3

There are three checkout counters at a local supermarket. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random and independently of the other. Let X denote the number of customers who choose counter 1 and Y the number of customers who select counter 2. Find the joint probability distribution of X and Y .

EXERCISE 4

Let X and Y denote the proportion of time, out of the workday, that employees I and II , respectively, actually spend performing their assigned tasks. The joint probability density function of X and Y is as follows:

$$f_{XY}(x, y) = \begin{cases} x + y & 0 \leq x \leq 1; 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

- Find $P(X < \frac{1}{2}, Y > \frac{1}{4})$. [Ans. $\frac{21}{64}$]
- Find $P(X + Y \leq 1)$. [Ans. $\frac{1}{3}$]

EXERCISE 5

A particular fast-food outlet is interested in the joint behavior of the random variables X , defined as the total time between a customer's arrival at the store and leaving the service window, and Y , the time that a customer waits in line before reaching the service window. Because X contains the time a customer waits in line, we must have $X \geq Y$. Suppose the joint probability density function of X and Y is as follows:

$$f_{XY}(x, y) = \begin{cases} e^{-x} & 0 \leq y \leq x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

with time measured in minutes.

- Find $P(X < 2, Y > 1)$. [Ans. $e^{-1} - 2e^{-2}$]
- Find $P(X \geq 2Y)$. [Ans. $\frac{1}{2}$]
- Find $P(X - Y \geq 1)$. Note that $X - Y$ denotes the time spent at the service window. [Ans. e^{-1}]

EXERCISE 6

Let X and Y have the joint probability density function given by

$$f_{XY}(x, y) = \begin{cases} kxy & 0 \leq x \leq 1; 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

- a. Find the constant k that makes this a probability density function. [Ans. 4]
- b. Find $P(X \leq \frac{1}{2}, Y \leq \frac{3}{4})$. [Ans. $\frac{9}{64}$]

EXERCISE 7

Refer to exercise 4.

- a. Find the marginal density functions for X and Y . [Ans. $f_X(x) = x + \frac{1}{2}$, $f_Y(y) = y + \frac{1}{2}$]
- b. Find $P(X \geq \frac{1}{2}/Y \geq \frac{1}{2})$. [Ans. $\frac{3}{5}$]
- c. If employee II spends exactly 50% of the day on assigned duties, find the probability that employee I spends more than 75% of the day on similar duties. In other words find $P(X > 0.75/Y = 0.5)$. [Ans. $\frac{11}{32}$]

EXERCISE 8

Refer to exercise 6.

- a. Find the marginal density functions of X and Y . [Ans. $f_X(x) = 2x$, $f_Y(y) = 2y$]
- b. Find $P(X \leq \frac{1}{2}/Y \geq \frac{3}{4})$. [Ans. $\frac{1}{4}$]
- c. Find the conditional density function of X given $Y = y$. [Ans. $2x$]
- d. Find the conditional density function of Y given $X = x$. [Ans. $2y$]
- e. Find $P(X \leq \frac{3}{4}/Y = \frac{1}{2})$. [Ans. $\frac{9}{16}$]