University of California, Los Angeles
Department of Statistics

Statistics 100A — Instructor: Nicolas Christou

Homework 6

EXERCISE 1
A coin is tossed 3 times independently. One of the variables of interest is the number of tails X. Let Y denote the amount of money won on a side bet in the following manner:
If the first tail occurs on the first toss, you win \$1.
If the first tail occurs on the second toss, you win \$2.
If the first tail occurs on the third toss, you win \$3.
If no tails appear you lose \$1.
Construct the joint probability distribution of X and Y. In other words complete the following table where the entries are the probabilities for each pair of values of the variables X and Y.

	X			
Y	0	1	2	3
-1	?	?	?	?
1	?	?	?	?
2	?	?	?	?
3	?	?	?	?

EXERCISE 2
A die is rolled and the number observed X is recorded. Then a coin is tossed number of times equal to the value of X. For example if $X = 2$ then the coin is tossed twice, etc. Let Y be the number of heads observed. Note: Assume that the die and the coin are fair.

a. Construct the joint probability distribution of X and Y.

b. Find the conditional expected value of Y given $X = 5$.

c. Find the conditional variance of Y given $X = 5$.

EXERCISE 3
There are three checkout counters at a local supermarket. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random and independently of the other. Let X denote the number of customers who choose counter 1 and Y the number of customers who select counter 2. Find the joint probability distribution of X and Y.

EXERCISE 4
Let X and Y denote the proportion of time, out of the workday, that employees I and II, respectively, actually spend performing their assigned tasks. The joint probability density function of X and Y is as follows:

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find $P(X < \frac{1}{2}, Y > \frac{1}{4})$. [Ans. $\frac{21}{64}$]

b. Find $P(X + Y \le 1)$. [Ans. $\frac{1}{3}$]

EXERCISE 5
A particular fast-food outlet is interested in the joint behavior of the random variables X, defined as the total time between a customer's arrival at the store and leaving the service window, and Y, the time that a customer waits in line before reaching the service window. Because X contains the time a customer waits in line, we must have $X \ge Y$. Suppose the joint probability density function of X and Y is as follows:

$$f_{XY}(x,y) = \begin{cases} e^{-x} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

with time measured in minutes.

a. Find $P(X < 2, Y > 1)$. [Ans. $e^{-1} - 2e^{-2}$]

b. Find $P(X \ge 2Y)$. [Ans. $\frac{1}{2}$]

c. Find $P(X - Y \ge 1)$. Note that $X - Y$ denotes the time spent at the service window. [Ans. e^{-1}]

EXERCISE 6
Let X and Y have the joint probability density function given by

$$f_{XY}(x,y) = \begin{cases} kxy & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the constant k that makes this a probability density function. [Ans. 4]

b. Find $P(X \le \frac{1}{2}, Y \le \frac{3}{4})$. [Ans. $\frac{9}{64}$]

EXERCISE 7
Refer to exercise 4.

a. Find the marginal density functions for X and Y. [Ans. $f_X(x) = x + \frac{1}{2}$, $f_Y(y) = y + \frac{1}{2}$]

b. Find $P(X \ge \frac{1}{2} | Y \ge \frac{1}{2})$. [Ans. $\frac{3}{5}$]

c. If employee II spends exactly 50% of the day on assigned duties, find the probability that employee I spends more than 75% of the day on similar duties. In other words find $P(X > 0.75 | Y = 0.5)$. [Ans. $\frac{11}{32}$]

EXERCISE 8
Refer to exercise 6.

a. Find the marginal density functions of X and Y. [Ans. $f_X(x) = 2x$, $f_Y(y) = 2y$]

b. Find $P(X \le \frac{1}{2} | Y \ge \frac{3}{4})$. [Ans. $\frac{1}{4}$]

c. Find the conditional density function of X given $Y = y$. [Ans. $2x$]

d. Find the conditional density function of Y given $X = x$. [Ans. $2y$]

e. Find $P(X \le \frac{3}{4} | Y = \frac{1}{2})$. [Ans. $\frac{9}{16}$]