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Statistics 100A **Instructor: Nicolas Christou**

Central limit theorem - Proof

If $X_1, X_2, \cdots, X_n$ are i.i.d. (independent and identically distributed) random variables having the same distribution with mean μ, variance σ^2, and moment generating function $M_X(t)$, then if $n \to \infty$ the limiting distribution of the random variable $Z = \frac{T-n\mu}{\sigma\sqrt{n}}$ (where $T = X_1 + X_2 + \cdots + X_n$) is the standard normal distribution $N(0,1)$.

Proof:

$$M_Z(t) = M_{\frac{T-n\mu}{\sigma\sqrt{n}}} = Ee^{\frac{T-n\mu}{\sigma\sqrt{n}}t} = e^{-\frac{n\mu}{\sigma\sqrt{n}}t} M_T(\frac{t}{\sigma\sqrt{n}})$$

But $T = X_1 + X_2 + \cdots + X_n$. From earlier discussion the mgf of the sum is equal to the product of the individual mgf. Here each X_i has mgf $M_X(t)$. Therefore,

$$M_T(\frac{t}{\sigma\sqrt{n}}) = \left[M_X(\frac{t}{\sigma\sqrt{n}})\right]^n$$

and so $M_Z(t)$ is equal to

$$M_Z(t) = e^{-\frac{n\mu}{\sigma\sqrt{n}}t} \left[M_X(\frac{t}{\sigma\sqrt{n}})\right]^n$$

One way to find the limit of $M_Z(t)$ as $n \to \infty$ is to consider the logarithm of $M_Z(t)$:

$$ln\ M_Z(t) = -\frac{\sqrt{n}\ \mu}{\sigma}t + n\ ln\ M_X(\frac{t}{\sigma\sqrt{n}})$$

Expanding $M_X(\frac{t}{\sigma\sqrt{n}})$, using the following (also see handout on mgf)

$$M_X(t) = \sum_x P(x) + \frac{t}{1!}\sum_x xP(x) + \frac{t^2}{2!}\sum_x x^2P(x) + \frac{t^3}{3!}\sum_x x^3P(x) + \cdots$$

we get

$$ln\ M_Z(t) = -\frac{\sqrt{n}\ \mu}{\sigma}t + n\ ln\left[1 + \frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{(\frac{t}{\sigma\sqrt{n}})^2}{2!}EX^2 + \frac{(\frac{t}{\sigma\sqrt{n}})^3}{3!}EX^3 + \cdots\right]$$

Now using the series expansion of $ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$ where $y = \frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{(\frac{t}{\sigma\sqrt{n}})^2}{2!}EX^2 + \frac{(\frac{t}{\sigma\sqrt{n}})^3}{3!}EX^3 + \cdots$ we get:

$$\begin{aligned} ln\ M_Z(t) = & - \frac{\sqrt{n}\ \mu}{\sigma}t + n\left[\frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{(\frac{t}{\sigma\sqrt{n}})^2}{2!}EX^2 + \frac{(\frac{t}{\sigma\sqrt{n}})^3}{3!}EX^3 + \cdots\right] \\ & - \frac{1}{2}\left[\frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{(\frac{t}{\sigma\sqrt{n}})^2}{2!}EX^2 + \frac{(\frac{t}{\sigma\sqrt{n}})^3}{3!}EX^3 + \cdots\right]^2 \\ & + \frac{1}{3}\left[\frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{(\frac{t}{\sigma\sqrt{n}})^2}{2!}EX^2 + \frac{(\frac{t}{\sigma\sqrt{n}})^3}{3!}EX^3 + \cdots\right]^3 - \cdots \end{aligned}$$

Factor out the powers of t we obtain:

$$\begin{aligned} ln\ M_Z(t) &= \left(-\frac{\sqrt{n}\ \mu}{\sigma} + \frac{\sqrt{n}\ EX}{\sigma}\right) t + \left(\frac{EX^2}{2\sigma^2} - \frac{(EX)^2}{2\sigma^2}\right) t^2 \\ &+ \left(\frac{EX^3}{6\sigma^3\sqrt{n}} - \frac{EX\ EX^2}{2\sigma^3\sqrt{n}} + \frac{(EX)^3}{3\sigma^3\sqrt{n}}\right) t^3 + \cdots \end{aligned}$$

Because $EX = \mu$ and $EX^2 - (EX)^2 = \sigma^2$ the last expression becomes

$$ln\ M_Z(t) = \frac{1}{2}t^2 + \left(\frac{EX^3}{6} - \frac{EX\ EX^2}{2} + \frac{(EX)^3}{3}\right) \frac{t^3}{\sigma^3\sqrt{n}} + \cdots$$

We observe that as $n \to \infty$ the limit of the previous expression is

$$\lim_{n\to\infty} ln\ M_Z(t) = \frac{1}{2}t^2$$

and therefore

$$\lim_{n\to\infty} M_Z(t) = e^{\frac{1}{2}t^2}.$$

But this is the mgf of the standard normal distribution. Therefore the limiting distribution of $\frac{T-n\mu}{\sigma\sqrt{n}}$ is the standard normal distribution $N(0,1)$.

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