University of California, Los Angeles
Department of Statistics

Statistics 100A **Instructor: Nicolas Christou**

Combinatorial analysis

Basic principle of counting:
Suppose two experiments are to be performed. Then, if the first experiment can result in m outcomes and if for each outcome of the first experiment there are n outcomes of the second experiment, then all together there are $m \times n$ possible outcomes.
Examples:

Permutations:
How many different *ordered* arrangements of the letters A, B, C are possible? There are 6 *permutations*: $ABC, ACB, BAC, BCA, CAB, CBA$. Or, using the basic principle of counting we can find the number of permutations as follows: $3 \times 2 \times 1 = 6$.
In general n objects can be ordered in $n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$ ways. Each arrangement it is called a *permutation*.
Example: In how many ways can 4 math books, 3 chemistry books, and 2 history books can be ordered so that books of the same subject are together?

Suppose k objects are to be selected and ordered from n objects ($k < n$).
Say, $n = 4$ (A, B, C, D), and $k = 3$. Let's list all the possible permutations:

$$\begin{array}{llll} ABC & BCD & CDA & DAB \\ ABD & BCA & CDB & DAC \\ ACB & BDA & CAB & DBC \\ ACD & BDC & CAD & DBA \\ ADB & BAC & CBD & DCA \\ ADC & BAD & CBA & DCB \end{array}$$

As we observe there are 24 permutations. Much easier, we can find the number of permutations using the basic principle of counting as follows: $4 \times 3 \times 2 = 24$.
In general, the number of ways that k objects can be selected and ordered from n objects are: $n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$. This can be simplified if we multiply and divide by $(n-k)!$:

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)\frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}.$$

Example: In how many ways can we select and order 5 cards from the 52 cards?

$$\frac{52!}{(52-5)!} = \frac{52!}{47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{47!} = 311875200.$$

Combinations:
Order does not count. For example $ABC, ACB, BAC, CAB, CBA, BCA$ are 6 permutations, but in terms of combinations they count for only one. For every 6 permutations we have one combination. In our example, if we divide 24 by 6 we get the number of combinations: $\frac{24}{6} = \frac{24}{3!} = 4$. In general: In how many ways can we select k objects from n objects:

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

A poker hand consists of 5 cards. How many poker hands are there?

$$\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2598960.$$

Sometimes we refer to $\binom{n}{k}$ as the *binomial coefficient* because it appears in the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

For example,
$(x+y)^4 = \binom{4}{0}x^0y^4 + \binom{4}{1}x^1y^3 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^3y^1 + \binom{4}{4}x^4y^0 = y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + x^4$.

Multinomial Coefficients

Suppose that n elements are to be divided into r groups ($r \geq 2$) in such a way that the j_{th} group contains n_j elements ($j = 1, \ldots, r$) with $\sum_{i=1}^{r} n_j = n$.

Example: Suppose 20 students are to be divided into three committees A, B, C in such a way that A has 8 members, B has 8 members, and C has 4 members. In how many ways can these 20 students can be assigned in the three committees.

Answer:
The n_1 elements in the first group can be selected from the total n available items in $\binom{n}{n_1}$ ways.
Then the n_2 elements of the second group can be selected from the remaining $n - n_1$ items in $\binom{n-n_1}{n_2}$ ways.
The n_3 elements of the third group can be selected from the remaining $n - n_1 - n_2$ items in $\binom{n-n_1-n_2}{n_3}$ ways.

$\ldots\ldots$

Finally, the n_r elements can be selected in $\binom{n-n_1-n_2-\ldots n_{r-1}}{n_r}$ ways.

Using the basic principle of counting the total number of ways is the multiplication of the above expressions:

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\ldots n_{r-1}}{n_r}$$

which can be simplified to

$$\frac{n!}{n_1!n_2!n_3!\ldots n_r!} \text{ and it is denoted with } \binom{n}{n_1, n_2, n_3, \ldots, n_r}.$$

The answer to the example above will be

$$\binom{20}{8, 8, 4} = \frac{20!}{8!8!4!} = 62355150.$$

Example:
In how many ways can the 52 cards of a standard deck of cards be divided into 4 piles (13 cards in each pile) such that the first pile contains 6 hearts, the second pile contains 4 hearts, the third pile contains 2 hearts, and the fourth pile contains 1 hearts.

Combinatorial analysis - some examples

(From: Mendenhall, W., Wackery, D.D, and Scheaffer, R.L. (1990), *Mathematical Statistics with Applications,* Fourth Edition. PWS-KENT Publishing Company, and
Sheldon Ross (2002), *A first Course in Probability*, Sixth Edition, Prentice Hall).

Example 1
From a group of 20 women and 16 men a committee consisting of 10 women and 10 men is to be formed. How many different committees are possible if

a. 2 of the women refuse to serve together?

b. 1 woman and 1 man refuse to serve together?

Example 2

a. The 52 cards of a standard 52-card deck are to be arranged randomly in a line. What is the probability that the ace of clubs ($A\clubsuit$) and the ace of spades ($A\spadesuit$) are next to each other?

b. What would be the above probability if the cards were randomly arranged in a circle?

Example 3
A committee of 3 is to be formed from a groupp of 20 people. How many different committees are possible? How many different committees if these 3 people will form a committee of president, vice-president, and treasurer?

Example 4
Consider $n-$digit numbers where each digit is one of the 10 integers $0, 1, \cdots, 9$.

a. How many such numbers are there for which no two consecutive digits are equal?

b. How many such numbers are there for which 0 appears as a digit a total of k times, $k = 0, 1, \cdots, n$?

Example 5
Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.

a. How many choices are possible?

b. How many choices are there in which all 3 students are in the same class?

c. How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?

d. How many choices are there in which all 3 students are in different classes?

Example 6
Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

Example 7
Consider a group of 6 children (3 boys and 3 girls).

a. In how many ways can the 3 boys and the 3 girls sit in a row?

b. In how many ways can the 3 boys and the 3 girls sit in a row if the boys and the girls must sit together?

c. In how many ways can these 6 children sit in a row if only the boys must sit together?

d. If these 6 children are to be seated in a row, what is the probability that no two children of the same sex are allowed to sit together?

Example 8
Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots \binom{n}{r}\binom{m}{0}.$$

Poker combinations

A poker hand consists of five cards. Players try for combinations of two or more cards of a kind, five-card sequences, or five cards of the same suit. Poker is played with a standard 52-card deck in which all suits are of equal value, the cards ranking from the ace high, downward through king, queen, jack, and the numbered cards 10 to the deuce. The ace may also be considered low to form a straight (sequence) ace through five as well as high with king-queen-jack-10.

Here is the traditional ranking of hands with some examples:

1. Royal Flush (ten-jack-queen-king-ace all of the same suit): $10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit$.
2. Straight Flush (five cards of the same suit in sequence other than royal flush): $2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit$.
3. Four of a kind, plus any fifth card: $5\clubsuit, 5\spadesuit, 5\diamondsuit, 5\heartsuit, 4\clubsuit$.
4. Full house (three of a kind plus a pair): $5\clubsuit, 5\spadesuit, 5\diamondsuit, 4\heartsuit, 4\clubsuit$.
5. Flush (any five cards of the same suit other than straight flush): $3\clubsuit, 6\clubsuit, 8\clubsuit, 10\clubsuit, J\clubsuit$.
6. Straight (five cards in sequence not all of the same suit): $2\spadesuit, 3\heartsuit, 4\spadesuit, 5\clubsuit, 6\diamondsuit$.
7. Three of a kind plus no pair: $5\clubsuit, 5\spadesuit, 5\diamondsuit, 4\heartsuit, 6\clubsuit$.
8. Two pairs plus any fifth card: $5\clubsuit, 5\spadesuit, 4\diamondsuit, 4\heartsuit, 9\clubsuit$.
9. One pair plus three other cards: $5\clubsuit, 5\spadesuit, 4\diamondsuit, 6\heartsuit, 10\clubsuit$.
10. Nothing

Rank	Poker Hand	# of Combinations
1.	Royal Flush	4
2.	Other Straight Flush	36
3.	Four of a kind	624
4.	Full House	3744
5.	Flush	5108
6.	Straight	10200
7.	Three of a kind	54912
8.	Two Pairs	123552
9.	One Pair	1098240
10.	Nothing	1302540
	Total	2598960

A standard 52-card deck consists of the following 13 denominations (4 cards in eac denomination), and 4 suits (13 cards in each suit):

$\clubsuit$	$\spadesuit$	$\diamondsuit$	$\heartsuit$
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9
10	10	10	10
J	J	J	J
Q	Q	Q	Q
K	K	K	K
A	A	A	A

There are $\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2598960$ different ways to choose 5 cards from the available 52 cards.

Let's compute the number of combinations of the following poker hand: four of kind plus any fifth card: We need 2 different denominations (for example 4 aces plus an eight). There are $\binom{13}{2}$ different ways to choose 2 denominations from the 13 available denominations. Now, there are $\binom{4}{4}$ ways to choose the 4 aces from the 4 aces, and $\binom{4}{1}$ different ways to choose one eight from the 4 eights. Also, the opposite could have occured (that is, 4 eights and 1 ace). Therefore the number of combinations of a four of a kind plus any fifth card is:

$$\binom{13}{2}\binom{4}{4}\binom{4}{1}2 = 624.$$

Similarly, we can find the number of combinations of a pair plus 3 other cards (for example 2 aces plus 1 two, 1 seven, 1 eight). We need 4 denominations. There are $\binom{13}{4}$ different ways to choose 4 denominations from the available 13 denominations. Now, there are $\binom{4}{2}$ different ways to choose a pair from one of the 4 denominations, and $\binom{4}{1}$ different ways to choose one card from each of the other 3 denominations. Also, we multiply by 4 because the pair could have come from the 2s or the 7s or the 8s. Therefore the number of combinations of a pair plus 3 other cards is:

$$\binom{13}{4}\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1}4 = 1098240.$$

For the examples above, to compute the probability that a player receives a four of kind plus any fifth card or a pair plus 3 other cards we simply divide the number of combinations that this particular poker hands occurs over the total number of ways in which 5 cards can be selected from the available 52 cards.

$$P(\text{four of a kind plus a fifth card}) = \frac{\binom{13}{2}\binom{4}{4}\binom{4}{1}2}{\binom{52}{5}} = \frac{624}{2598960} = 0.00024.$$