

Poisson, Gamma, and Exponential distributions

• **A. Relation of Poisson and exponential distribution:**

Suppose that events occur in time according to a Poisson process with parameter  $\lambda$ . So  $X \sim Poisson(\lambda)$ . Let  $T$  denote the length of time until the first arrival. Then  $T$  is a continuous random variable. To find the probability density function (pdf) of  $T$  we begin with the cumulative distribution function (cdf) of  $T$  as follows:

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(X = 0)$$

In words: The probability that we observe the first arrival after time  $t$  is the same as the probability that we observe no arrivals from now until time  $t$ . But  $X$  is Poisson with parameter  $\lambda$  which has parameter  $\lambda t$  over the time interval  $(0, t)$ . We compute the above using:

$$F(t) = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \Rightarrow F(t) = 1 - e^{-\lambda t}.$$

To find the pdf of  $T$  we take the derivative of the cdf w.r.t.  $t$  to get:

$$f(t) = F(t)' = \lambda e^{-\lambda t}.$$

We observe that if  $X \sim Poisson(\lambda)$  the time until the first arrival is exponential with parameter  $\lambda$ .

Example:

Suppose that an average of 20 customers per hour arrive at a shop according to a Poisson process ( $\lambda = \frac{1}{3}$  per minute). What is the probability that the shopkeeper will wait more than 5 minutes before the first customer arrives?

• **B. Relation of Poisson and gamma distribution:**

Suppose that events occur in time according to a Poisson process with parameter  $\lambda$ . So  $X \sim \text{Poisson}(\lambda)$ . Let  $T$  denote the length of time until  $k$  arrivals. Then  $T$  is a continuous random variable. To find the probability density function (pdf) of  $T$  we begin with the cumulative distribution function (cdf) of  $T$  as follows:

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(X < k) = 1 - P(X \leq k - 1)$$

In words: The probability that we observe the  $k_{th}$  arrival after time  $t$  is the same as the probability that we observe less than  $k$  arrivals from now until time  $t$ . But  $X$  is Poisson with parameter  $\lambda$  which has parameter  $\lambda t$  over the time interval  $(0, t)$ . We compute the above using:

$$F(t) = 1 - P(X \leq k - 1) = 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!} = 1 - e^{-\lambda t} \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!}$$

To find the pdf of  $T$  we take the derivative of the cdf w.r.t.  $t$  to get:

$$\begin{aligned} f(t) &= F(t)' \\ &= e^{-\lambda t} \lambda \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - e^{-\lambda t} \sum_{x=0}^{k-1} \frac{x(\lambda t)^{x-1} \lambda}{x!} \\ &= e^{-\lambda t} \lambda \left[ \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{x=1}^{k-1} \frac{x(\lambda t)^{x-1}}{x!} \right] \\ &= e^{-\lambda t} \lambda \left[ \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{x=1}^{k-1} \frac{x(\lambda t)^{x-1}}{x(x-1)!} \right] \\ &= e^{-\lambda t} \lambda \left[ \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{x=1}^{k-1} \frac{(\lambda t)^{x-1}}{(x-1)!} \right], \text{ for the second term let } y = x - 1 \\ &= e^{-\lambda t} \lambda \left[ \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{y=0}^{k-2} \frac{(\lambda t)^y}{y!} \right]. \end{aligned}$$

The square bracket is reduced to  $\frac{(\lambda t)^{k-1}}{(k-1)!}$  because,

$$\begin{aligned} &\left[ \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{y=0}^{k-2} \frac{(\lambda t)^y}{y!} \right] = \\ &1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \dots + \frac{(\lambda t)^{k-2}}{(k-2)!} + \frac{(\lambda t)^{k-1}}{(k-1)!} - \\ &1 - \frac{(\lambda t)}{1!} - \frac{(\lambda t)^2}{2!} - \frac{(\lambda t)^3}{3!} - \dots - \frac{(\lambda t)^{k-2}}{(k-2)!}. \end{aligned}$$

So far we have

$$f(t) = e^{-\lambda t} \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} = \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!}$$

But, since  $k$  is an integer (number of  $k$  arrivals),  $\Gamma(k) = (k-1)!$ . The expression above can be written as:

$$f(t) = \frac{t^{k-1} \lambda^k e^{-\lambda t}}{\Gamma(k)}$$

Compare  $f(t)$  with the gamma pdf:

$$f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, \quad \alpha, \beta > 0, x \geq 0.$$

We observe that  $f(t)$  is the density of a gamma distribution with parameters  $\alpha = k$  and  $\beta = \frac{1}{\lambda}$ .

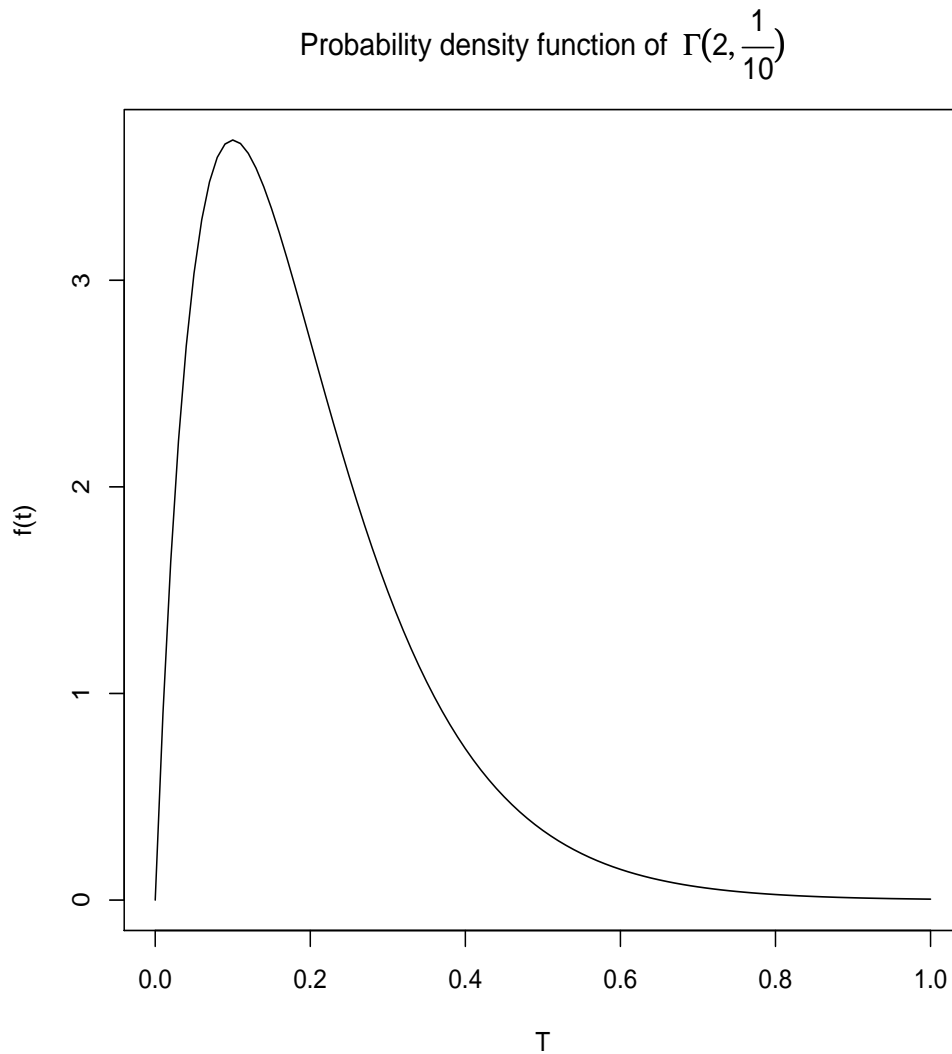
Conclusion: If  $X \sim \text{Poisson}(\lambda)$  the time until  $k$  arrivals is  $\Gamma(k, \frac{1}{\lambda})$ .

Example:

Suppose customers arrive at a store as a Poisson process with  $\lambda = 10$  customers per hour.

- a. What is the distribution of the time until the second customer arrives (see graph on next page)?
- b. Find the probability that one has to wait at least half an hour until the second customer arrives.

Part (a):



The graph above was constructed in R:

```
> t <- seq(0,1,0.01)
> ft <- 100*t*exp(-10*t)
> plot(t,ft,type="l", xlab="T", ylab="f(t)")
> title(main=expression(paste("Probability density function of ",
  Gamma(2,frac(1,10))))))
```