Problem 1  (25 points)
Answer the following questions:

a. Let $U$ be a random variable with moment generating function $M_U(t) = e^{500t + 5000t^2}$. Use the normal distribution to compute $P(27100 < (U - 500)^2 < 50200)$.

b. Let $X = (X_1, \ldots, X_n)'$ be a random vector with joint moment generating function $M_X(t)$. Let $W = \sum_{i=1}^{n} a_i X_i + b$ and $U = \sum_{i=1}^{n} c_i X_i + d$. Show that the joint moment generating function of $W$ and $U$ is $M_{W,U}(s,r) = e^{bs+dr} M_X(a_1 s + c_1 r + \ldots + a_n s + c_n r)$. 

c. If $Y_1, \ldots, Y_n$ are i.i.d. random variables with $Y_i \sim N(\mu, \sigma)$, prove that $\bar{Y}$ is independent of $\sum_{i=1}^{n-1} (Y_i - Y_{i+1})^2$.
Note: If $\bar{Y}$ is independent of the differences given in the summation then it is independent of any function of them.

d. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli($p$). Suppose $q = \left( \frac{\sum_{i=1}^n X_i}{n} \right)^2$. Find the expected value of $q$. 
Problem 2  (25 points)
Answer the following questions:

a. Suppose \( X \sim b(n, p) \) and \( Y \sim b(m, p) \). Using properties of moment generating functions we showed that \( X + Y \sim b(n + m, p) \). A different approach is to show this result analytically by expanding \( P(X + Y = k) = \sum_{i=0}^{n} P[X = i, Y = k - i] \). You will need the following combinatorial identity \( \sum_{i=0}^{n} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k} \).

b. Suppose \( Z_1, \ldots, Z_n \) are i.i.d. Bernoulli random variables with probability of success \( p \). Let \( W \) be a symmetric \( n \times n \) matrix of constants such that \( w_{ii} = 0, i = 1, \ldots, n \). For example, the entries of \( W \) can be 0 or 1, where 1 denotes connectivity between neighboring counties and 0 no connectivity. Find \( E[Z'WZ] \).
c. Refer to homework 3, exercise 4. Suppose \( m \) points are randomly selected. Find the pdf of \( \sum_{i=1}^{m} R_i^3 \), where \( R_i \) is the distance from point \( i \) to the nearest particle. Find \( E \left( \left[ \sum_{i=1}^{m} R_i^3 \right]^{1/3} \right) \).

d. We discussed in class the multinomial probability distribution and its joint moment generating function. Here is a note on the multinomial distribution: A sequence of \( n \) independent experiments is performed and each experiment can result in one of \( r \) possible outcomes with probabilities \( p_1, p_2, \ldots, p_r \) with \( \sum_{i=1}^{r} p_i = 1 \). Let \( X_i \) be the number of the \( n \) experiments that result in outcome \( i \), \( i = 1, 2, \ldots, r \). Then, \( P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r) = \frac{n!}{n_1!n_2!\cdots n_r!} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r} \). The joint moment generating function of the multinomial distribution is given by \( M_X(t) = (p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_r e^{t_r})^n \). Consider \( D = \frac{\bar{X}_1}{n} - \frac{\bar{X}_2}{n} \). This is the difference in the two sample proportions. (The corresponding difference in the population proportions is \( p_1 - p_2 \).) Find \( E[D] \) and \( \text{var}[D] \).
Problem 3  (25 points)
Answer the following questions:

a. Suppose a random variable $X$ has the pdf

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}, \quad 0 < x < \infty.$$  

Find the mean and variance of $X$ without integration. You will need to start with $E[X] = \int_{x} x f(x) dx$ and evaluate this without integration.

b. Refer to question (a). Find the transformation $Y = g(X)$ so that $Y \sim \Gamma(\alpha, \beta)$. 


c. Let $X \sim \text{beta}(\alpha, \beta)$ and $Y \sim \text{beta}(\alpha + \beta, \gamma)$. It is given that $X$ and $Y$ are independent. Let $U = XY$ and $V = X$. Find the joint pdf of $U$ and $V$ and use it to show that the marginal pdf of $U$ is a beta distribution. What are the parameters of this beta distribution. Note: $0 < x < 1, 0 < y < 1$ and $0 < u < v < 1$. You will need the last inequality when you find the pdf of $U$.

d. Suppose $X$ and $Y$ are independent random variables with $X \sim N(\mu, \sigma)$ and $Y \sim N(\gamma, \sigma)$. Consider $U = X + Y$ and $V = X - Y$. Show that $U$ and $V$ are independent using all the different methods we discussed on independence.
Problem 4  (25 points)
Answer the following questions:

a. Suppose $X, Y$ have the joint probability density function given by
\[ f(x, y; \theta) = e^{-\theta x - \frac{y}{\theta}}, \quad x > 0, y > 0. \]

Consider the i.i.d sample $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Find the mean and variance of $\sqrt{\sum Y_i \sum X_i}$.

b. Assume that $Y_1, \ldots, Y_n$ follow the multivariate normal distribution with the same mean $\mu$, variance $\sigma^2$, and $\text{cov}(Y_i, Y_j) = \rho \sigma^2$. Show that $\text{var}(\bar{Y}) = \frac{\sigma^2}{n} [1 + (n - 1) \rho]$. What is the distribution of $\bar{Y}$? What result did you use to find the distribution of $\bar{Y}$?
c. Answer the following questions:

1. Let $Y_1, Y_2, \ldots, Y_{16}$ i.i.d. $N(5, 2)$. Write the distribution of the following:
   $\mathbf{Y}$ (vector)
   
   $\frac{Y_1 - 5}{2}$
   
   $\frac{Y - 5}{2}$
   
   $\sum_{i=1}^{16} Y_i$
   
   $\bar{Y}$
   
   $Y_1 - 2Y_2$.

2. Let $X$ be a random variable with pdf $f(x)$. Evaluate the following integrals without integration! Please explain.
   1. $\int_0^\infty x^{\frac{1}{2}} e^{-\frac{x}{2}} dx$.
   2. $\int_0^1 x^{\frac{1}{2}} (1 - x)^{\frac{1}{2}} dx$.
   3. $\int_{-\infty}^{\infty} e^{-\frac{(x-3)^2}{8}} dx$.

   d. If $\mathbf{Y}$ and $Y$ are independent random variables, show that $E(Y^{k}) = E_{E \mathbf{Y}^{k}}$. Apply this result to example 2 of handout #10. Note is $Q_1$ and $Q_2$ are independent then $E_{Q_1}Q_2 = E_{Q_1}E{Q_2}$ and this holds for any functions of $Q_1$ and $Q_2$. 

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