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Statistics 100B **Instructor: Nicolas Christou**

Homework 10 - Solutions

Exercise 1
Here we are testing:
$H_0 : p = \frac{1}{2}$
$H_a : p \neq \frac{1}{2}$

a. $\alpha = P(X = 0 \text{ } or \text{ } X = 10) = \binom{10}{0} \frac{1}{2}^0 \frac{1}{2}^{10} + \binom{10}{10} \frac{1}{2}^{10} \frac{1}{2}^0 = 0.0020$. Therefore the significance level is $\alpha = 0.0020$.

b. $1 - \beta = P(X = 0 \text{ } or \text{ } X = 10 | p = 0.10) = \binom{10}{0} 0.1^0 0.9^{10} + \binom{10}{10} 0.1^{10} 0.9^0 = 0.3487$. Therefore the power of the test is $1 - \beta = 0.3487$.

Exercise 2
Part (a). We follow the 4 steps:

1.
$$H_0 : \quad \mu = 130$$
$$H_a : \quad \mu < 130$$

2. We compute the test statistic z:
$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{128.6 - 130}{\frac{3}{\sqrt{40}}} \Rightarrow z = -2.95$$

3. We find the rejection region. Here we use significance level $\alpha = 0.05$, therefore the rejection region will be when $z < -1.645$.

4. Conclusion: Since $z = -2.95 < -1.645$ we reject H_0. Therefore based on the evidence the mean output voltage is less than 130 volts.

Part (b). Yes, it is possible that the mean output voltage is still 130 volts. We may have commited a type I error.

Part(c). $1 - \beta = P(\bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} | \mu = \mu_a) = P(z < \frac{\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}}) = P(z < \frac{130 - 1.645 \frac{3}{\sqrt{40}} - 128.6}{\frac{3}{\sqrt{40}}}) = P(z < 1.31) = 0.9049$.
Therefore the Type II error is $\beta = 0.0951$.

Part(d).
a. If we decrease the type I error α then the type II error β will increase.

b. If the true population mean is 129.6 volts the type II error will increase (the hypothesized mean is very close to the true mean and therefore it is more difficult to detect a small difference).

Exercise 3
Answer the following questions:

a. The lifetime of certain batteries are supposed to have a variance of 150 hours^2. Using $\alpha = 0.05$ test the following hypothesis
$H_0 : \sigma^2 = 150$
$H_a : \sigma^2 > 150$
if the lifetimes of 15 of these batteries (which constitutes a random sample from a normal population) have:
$$\sum_{i=1}^{15} x_i = 250, \quad \sum_{i=1}^{15} x_i^2 = 8000.$$
where X denotes the lifetime of a battery.

Answer:
We compute first the sample variance s^2:
$$s^2 = \frac{1}{n-1} \left[\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i \right)^2}{n} \right] = \frac{1}{14} \left(8000 - \frac{250^2}{15} \right) = 273.8.$$

The test statistic is: $\frac{(n-1)s^2}{\sigma_0^2} = \frac{(15-1)273.8}{150} = 25.55$. The critical value is $\chi^2_{0.05;14} = 23.68$, therefore H_0 is rejected.

b. A confidence interval is *unbiased* if the expected value of the interval midpoint is equal to the estimated parameter. For example the midpoint of the interval $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is $\bar{x}$, and $E(\bar{x}) = \mu$. Now consider the confidence interval for σ^2. Show that the expected value of the midpoint of this confidence interval is not equal to σ^2.

Answer:
The midpoint of the confidence interval for σ^2 is:

$$\frac{1}{2}\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}\right]$$

and its expected value is:

$$\frac{1}{2}E\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}\right] = \frac{1}{2}\left[\frac{(n-1)E(s^2)}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)E(s^2)}{\chi^2_{\frac{\alpha}{2};n-1}}\right].$$

Since $E(s^2) = \sigma^2$ the expression above is:

$$\frac{1}{2}\sigma^2\left[\frac{(n-1)}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)}{\chi^2_{\frac{\alpha}{2};n-1}}\right] \neq \sigma^2$$

because

$$\left[\frac{(n-1)}{\chi^2_{1-\frac{\alpha}{2};n-1}} + \frac{(n-1)}{\chi^2_{\frac{\alpha}{2};n-1}}\right] \neq 2.$$

Exercise 4
Let X be a uniform random variable on $(0, \theta)$. You have exactly one observation from this distribution and you want to test the null hypothesis $H_0 : \theta = 10$ against the alternative $H_a : \theta > 10$, and you want to use significance level $\alpha = 0.10$. Two testing procedures are being considered:
Procedure G rejects H_0 if and only if $X \geq 9$.
Procedure K rejects H_0 if either $X \geq 9.5$ or if $X \leq 0.5$.

a. Confirm that Procedure G has a Type I error probability of 0.10.
Answer:

$$\alpha = P(X \geq 9|\theta = 10) \int_9^{10} \frac{1}{10}dx = \frac{x}{10}|_9^{10} = 0.10.$$

b. Confirm that Procedure K has a Type I error probability of 0.10.
Answer:

$$\alpha = P(X \geq 9.5|\theta = 10) + P(X \leq 0.5|\theta = 10) = \int_{9.5}^{10} \frac{1}{10}dx + \int_0^{0.5} \frac{1}{10}dx = \frac{x}{10}|_{9.5}^{10} + \frac{x}{10}|_0^{0.5} = 0.10.$$

c. Find the power of Procedure G when $\theta = 12$.
Answer:

$$1 - \beta = P(X \geq 9|\theta = 12) \int_9^{12} \frac{1}{12}dx = \frac{x}{12}|_9^{12} = 0.25.$$

d. Find the power of Procedure K when $\theta = 12$.
Answer:

$$1 - \beta = P(X \geq 9.5|\theta = 12) + P(X \leq 0.5|\theta = 12) = \int_{9.5}^{12} \frac{1}{12}dx + \int_0^{0.5} \frac{1}{12}dx = \frac{x}{12}|_{9.5}^{12} + \frac{x}{12}|_0^{0.5} = 0.25.$$

Exercise 5
Suppose that the length in millimeters of metal fibers produced by a certain process follow the normal distribution with mean μ and standard deviation σ (both are unknown). We will test:
$H_0 : \mu = 5.2$
$H_a : \mu \neq 5.2$
A sample size of $n = 15$ metal fibers was selected and was found that $\bar{x} = 5.4$ and $s = 0.4266$.

a. Approximate the $p-$value using only your t table and use it to test this hypothesis. Assume $\alpha = 0.05$.
Answer:
Compute the t statistic:

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{5.4 - 5.2}{\frac{0.4266}{\sqrt{15}}} = 1.816.$$

From the t table we approximate the p-value as: p-value$= 2 \times P(t_{14} > 1.816)$. We conclude that the p-value is between 5% and 10%. Therefore, H_0 is not rejected.

b. Assume now that the population standard deviation is known and it is equal to $\sigma = 0.4266$. Compute the power of the test when the actual mean is $\mu_a = 5.35$ and you can accept $\alpha = 0.05$.
Answer:
This is two-sided test, therefore the power is computed as follows:

$$\begin{aligned}
1-\beta &= P(\bar{x} > \mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}|\mu=\mu_a) + P(\bar{x} < \mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}|\mu=\mu_a) \\
&= P(z > \frac{\mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}}) + P(z < \frac{\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}}) \\
&= P(z > \frac{5.2 + 1.96\frac{0.4266}{\sqrt{15}} - 5.35}{\frac{0.4266}{\sqrt{15}}}) + P(z > \frac{5.2 - 1.96\frac{0.4266}{\sqrt{15}} - 5.35}{\frac{0.4266}{\sqrt{15}}}) \\
&= P(z > 0.60) + P(z < -3.32) = (1-0.7257) + (1-0.9995) = 0.2748.
\end{aligned}$$

c. On the previous page draw the two distributions (under H_0 and under H_a) and show the Type I error and the Type II error on them.
Answer:
This is similar to the one-sided examples we did in class. Here we have a two-sided test.

d. Assume now that the hypothesis we are testing is
$H_0 : \mu = 5.2$
$H_a : \mu > 5.2$
Determine the sample size needed in order to detect with probability 95% a shift from $\mu_0 = 5.2$ to $\mu_a = 5.3$ if you are willing to accept a Type I error $\alpha = 0.05$. Assume $\sigma = 0.4266$.
Answer:

$$n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_a - \mu_0)^2} = \frac{(1.645 + 1.645)^2 0.4266^2}{(5.3-5.2)^2} = 196.99 \approx 197.$$