HW 2 100B - SOLUTIONS

EXERCISE 1: $X \sim N(\mu, \sigma) \to M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

(a). $M_{aX+b}(t) = E\, e^{t(aX+b)}$

$$= e^{tb}\, E\, e^{taX} = e^{tb} M_X(at)$$

$$= e^{tb}\, e^{ta\mu + \frac{1}{2}t^2a^2\sigma^2}$$

$$= e^{t(a\mu+b) + \frac{1}{2}t^2a^2\sigma^2} \quad \therefore\ aX+b \sim N(a\mu+b, a\sigma)$$

(b). CDF: Let $Y = aX+b$

$$F_Y(y) = P(Y \le y) = P(aX+b \le y)$$

$$= P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) = \frac{1}{a}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(\frac{y-b}{a}-\mu\right)^2}$$

$$= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2a^2\sigma^2}\left(y-(a\mu+b)\right)^2}$$

AGAIN WE SEE THAT $aX+b \sim N(a\mu+b, a\sigma)$

EXERCISE 2 :

(a). $\ln X \sim N(\mu, \sigma)$

$M_{\ln X}(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$ (BECAUSE $\ln X$ FOLLOWS $N(\mu, \sigma)$)

BUT ALSO

$M_{\ln X}(t) = Ee^{t\ln X} = Ee^{\ln X^t} = EX^t$

THEREFORE, $EX^t = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

SET $t = 1$ TO GET $EX = e^{\mu + \frac{1}{2}\sigma^2}$

SET $t = 2$ TO GET $EX^2 = e^{2\mu + 2\sigma^2}$

AND $\mathrm{VAR}(X) = EX^2 - (EX)^2$

$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$

$= e^{2\mu + \sigma^2}\left[e^{\sigma^2} - 1\right]$

(b). $M_Y(t) = M_{\sum k_i X_i}(t)$

$$= E\, e^{t\sum k_i X_i} = E e^{tk_1X_1} \cdot E e^{tk_2X_2} \cdots E e^{tk_nX_n}$$

$$= M_{X_1}(tk_1) \cdot M_{X_2}(tk_2) \cdots M_{X_n}(tk_n)$$

$$= e^{tk_1\mu_1 + \frac{1}{2}t^2k_1^2\sigma_1^2} \cdot e^{tk_2\mu_2 + \frac{1}{2}t^2k_2^2\sigma_2^2} \cdots e^{tk_n\mu_n + \frac{1}{2}t^2k_n^2\sigma_n^2}$$

$$= e^{t\sum k_i\mu_i + \frac{1}{2}t^2\sum k_i^2\sigma_i^2}$$

$\therefore\ Y \sim N\left(\sum k_i\mu_i,\ \sqrt{\sum k_i^2\sigma_i^2}\right)$

(c). $X_i \overset{iid}{\sim} \Gamma(\alpha, \beta)$

$$M_{X_i}(t) = (1-\beta t)^{-\alpha}$$

$$M_T(t) = M_{X_1+\cdots+X_n}(t)$$

$$= \left(M_{X_i}(t)\right)^{n}$$

$$= (1-\beta t)^{-n\alpha}$$

$\therefore T \sim \Gamma(n\alpha, \beta)$

AND $M_{\bar{X}}(t) = M_{\frac{T}{n}}(t) = M_T\left(\frac{t}{n}\right)$

$$= \left(1-\beta \frac{t}{n}\right)^{-n\alpha}$$

$\therefore \bar{X} \sim \Gamma\left(n\alpha, \frac{\beta}{n}\right)$

EXERCISE 3 :

$X \sim N(\mu, \sigma)$ STEIN'S LEMMA

$$E\left(g(X)(X-\mu)\right) = \sigma^2 E g'(X)$$

$$E X^4 = E X^3 (X-\mu+\mu)$$

$$= E X^3 (X-\mu) + \mu E X^3$$

$$= \sigma^2 E 3X^2 + \mu E X^2 (X-\mu+\mu)$$

$$= 3\sigma^2 E X^2 + \mu E X^2 (X-\mu) + \mu^2 E X^2$$

$$= 3\sigma^2 (\mu^2+\sigma^2) + \mu E 2X + \mu^2 (\mu^2+\sigma^2)$$

$$= (3\sigma^2+\mu^2)(\mu^2+\sigma^2) + 2\mu^2 = \ldots$$

ETC.

EXERCISE 4 :

$$\begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix} \equiv \underset{\sim}{X} - \underset{\sim}{1}\bar{X}$$

WHERE $\underset{\sim}{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

WRITE $\bar{X} = \frac{1}{n} \underset{\sim}{1}' \underset{\sim}{X}$

THEN $\underset{\sim}{X} - \underset{\sim}{1}\bar{X} = \underset{\sim}{X} - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{X}$

$$= \left(I - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}' \right) \underset{\sim}{X} = A\underset{\sim}{X}$$

SINCE $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma) \rightarrow \left. \begin{array}{l} E\underset{\sim}{X} = \mu \underset{\sim}{1} \\ VAR(\underset{\sim}{X}) = \sigma^2 I \end{array} \right\}$

$$EAX \equiv AEX = \mu \left(I - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}' \right) \underset{\sim}{1} = \underset{\sim}{0}.$$

$$VAR(AX) = \sigma^2 AA' = \sigma^2 \left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}' \right)\left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}' \right)$$

$$= \sigma^2 \left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}' \right)$$

TYPICAL ELEMENTS:

$VAR(X_i - \bar{X}) = \sigma^2 \left(1 - \frac{1}{n}\right)$

$COV(X_i - \bar{X}, X_j - \bar{X}) = -\frac{\sigma^2}{n}$

EXERCISE 5.

(a) $X \sim U(0,1)$ $\quad EX = \frac{1}{2}, \quad \operatorname{var}(X) = \frac{1}{12}.$

$$EX^2 = \sigma^2 + \mu^2 = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}.$$

$$\therefore E\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.$$

$\operatorname{var}(X) = \frac{1}{12},$

$$\operatorname{var}(X^2) = EX^4 - (EX^2)^2$$
$$= \int_0^1 x^4\,dx - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

$$\operatorname{Cov}(X, X^2) = EX^3 - (EX)(EX^2) = \int_0^1 x^3\,dx - \frac{1}{2}\cdot\frac{1}{3}$$
$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

$$\therefore \operatorname{var}\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{4}{45} \end{pmatrix}$$

PART (b): $X_1 \sim \Gamma(\alpha, 1)$, $X_2 \sim \Gamma(\alpha+\frac{1}{2}, 1)$

IF $X \sim \Gamma(\alpha, \theta)$

THEN $EX^K = \dfrac{\Gamma(\alpha+K)\,\theta^K}{\Gamma(\alpha)}$

$Y = 2\sqrt{X_1 X_2}$

$EY = 2E\sqrt{X_1 X_2}$

$= 2 E X_1^{1/2} \, E X_2^{1/2}$

$= 2 \dfrac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} 1^{1/2} \cdot \dfrac{\Gamma(\alpha+\frac{1}{2}+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})} 1^{1/2} = \dfrac{2\Gamma(\alpha+1)}{\Gamma(\alpha)}$

$= 2\alpha.$

$VAR(Y) = VAR\left(2\sqrt{X_1 X_2}\right)$

$= 4\, VAR\sqrt{X_1 X_2}$

$= 4\left(E(X_1 X_2) - \left(E\sqrt{X_1 X_2}\right)^2\right)$

$= 4(EX_1 \, EX_2 - \ldots$

EXERCISE 6

Problem 4 (25 points)

Answer the following questions:

a. Let $\mathbf{X} = (X_1, \dots, X_n)'$ be a random vector with joint moment generating function $M_{\mathbf{X}}(\mathbf{t})$. In class we discuss this theorem: Let $M_i(\mathbf{t}) = \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i}$, $M_{ii}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i^2}$, and $M_{ij}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i \partial t_j}$. Then, $EX_i = M_i(\mathbf{0})$, $EX_i^2 = M_{ii}(\mathbf{0})$, and $EX_iX_j = M_{ij}(\mathbf{0})$. Prove this theorem when $n = 2$.

$$M_{\underline{X}}(\underline{t}) = E\, e^{t_1 X_1 + t_2 X_2} = \iint e^{t_1 x_1 + t_2 x_2} f(x_1, x_2)\, dx_1\, dx_2$$

$$M_1(\underline{t}) = \frac{\partial M_{\underline{X}}(\underline{t})}{\partial t_1} = E\, X_1 \left(e^{t_1 X_1 + t_2 X_2}\right) = E X_1 \quad (\text{when } t_1 = 0,\ t_2 = 0)$$

$$M_{11}(\underline{t}) = \frac{\partial^2 M_X(\underline{t})}{\partial t_1^2} = E\, X_1^2 \left(e^{t_1 X_1 + t_2 X_2}\right) = X_1^2 \quad (\text{when } t_1 = 0,\ t_2 = 0)$$

$$M_{12}(\underline{t}) = \frac{\partial^2 M_X(\underline{t})}{\partial t_1 \partial t_2} = E\, X_1 X_2 \left(e^{t_1 X_1 + t_2 X_2}\right) = E X_1 X_2$$

$(\text{when } t_1 = 0,\ t_2 = 0)$

EXERCISE 6

(b).

Problem 1 (25 points)
Answer the following questions:

$f(u) = \frac{u^{\alpha-1} e^{-u/\beta}}{\Gamma(\alpha)\beta^\alpha}$

a. Suppose $U \sim \Gamma(\alpha, \beta)$, with $\alpha > 0, \beta > 0$ and let $Y = e^U$. Find the probability density function of Y.

$$F_Y(y) = P(Y \le y) = P(e^U \le y) = P(U \le \ln y) = F_U(\ln y)$$

$$f(y) = \frac{1}{y} f_U(\ln y) = \frac{1}{y} \frac{(\ln y)^{\alpha-1} e^{-\ln y/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} (\ln y)^{\alpha-1} \cdot y^{1/\beta - 1}$$

b. Refer to part (a). Find EY and $var(Y)$.

$$M_U(t) = Ee^{tU} = (1-\beta t)^{-\alpha}$$

$$t=1 \to Ee^U = EY = (1-\beta)^{-\alpha}$$

$$t=2 \to Ee^{2U} = EY^2 = (1-2\beta)^{-\alpha}$$

$$var(Y) = (1-2\beta)^{-\alpha} - (1-\beta)^{-2\alpha}$$

Exercise 7

$$f(y) = \theta y^{\theta-1} \qquad 0<y<1 \;, \quad \theta>0$$

$$W = -\ln(Y) \qquad F_W(w) = P(W \le w)$$

$$= P(-\ln Y \le w) = P(\ln y \geq -w)$$

$$= P(Y \geq e^{-w}) = 1 - P(Y \le e^{-w}) = 1 - F_Y(e^{-w})$$

$$\therefore f(w) = e^{-w}\,\theta\, e^{-w(\theta-1)} = \theta e^{-\theta w}$$

$$\therefore \quad W \sim \exp(\theta).$$

$$M_{2\theta \sum w_i}(t) = M_{2\theta w_1}(t) \dots M_{2\theta w_n}(t)$$

$$= \left(M_{w_i}(2\theta t)\right)^n = \left(1 - \frac{2\theta t}{\theta}\right)^{-n}$$

$$= (1-2t)^{-n}$$

NOTE: FROM (d) $M_{w_i}(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$

$$\therefore \quad 2\theta \sum w_i \sim \Gamma(n, 2).$$

Exercise 8

$X \sim N(\mu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} \quad e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x}$$

Let $h(x) = 1$

$c(\underline{\theta}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}}$

$w_1(\underline{\theta}) = \frac{1}{\sigma^2}$, $\quad t_1(x) = -\frac{x^2}{2}$

$w_2(\underset{\sim}{\theta}) = \frac{\mu}{\sigma^2}$, $\quad t_2(x) = x$