Homework 4 solutions

EXERCISE 1:

$$M_{\bar{X},\bar{Y}}(t_1,t_2) = E\, e^{t_1\bar{X} + t_2\bar{Y}}$$

$$= E\, e^{t_1 \frac{X_1+\cdots+X_n}{n} + t_2 \frac{Y_1+\cdots+Y_n}{n}}$$

$$= \left[E\, e^{\frac{t_1}{n}X_1 + \frac{t_2}{n}Y_1}\right] \cdots \left[E\, e^{\frac{t_1}{n}X_n + \frac{t_2}{n}Y_n}\right]$$

(BECAUSE (X_i, Y_i) $i=1,2,\ldots,n$ ARE INDEPENDENT)

$$= \Big\{ \underbrace{M_{X_i,Y_i}\left(\frac{t_1}{n}, \frac{t_2}{n}\right)}_{} \Big\}^n$$

THIS IS THE JOINT MGF OF BIVARIATE NORMAL

$$= \left(e^{\frac{\underset{\sim}{t}'}{n}\underset{\sim}{\mu} + \frac{1}{2}\frac{\underset{\sim}{t}'}{n}\underset{\sim}{\Sigma}\frac{\underset{\sim}{t}}{n}} \right)^n$$

WHERE $\underset{\sim}{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$

$$= e^{\underset{\sim}{t}'\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}'\frac{\underset{\sim}{\Sigma}}{n}\underset{\sim}{t}} \quad \therefore \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2\left(\underset{\sim}{\mu}, \frac{\underset{\sim}{\Sigma}}{n}\right)$$

Exercise 2

(a)

$$W = X - \mu_1$$

$$Q = (Y - \mu_2) - \rho \frac{\sigma_2}{\sigma_1}(X - \mu_1)$$

$\mathrm{Var}(Q) = \sigma_2^2 + \rho^2 \frac{\sigma_2^2}{\sigma_1^2}\sigma_1^2 - 2\rho\frac{\sigma_2}{\sigma_1}\rho\sigma_1\sigma_2 = \sigma_2^2(1-\rho^2)$.

$$\begin{pmatrix} W \\ Q \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho\frac{\sigma_2}{\sigma_1} & 1 \end{bmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{bmatrix} -\mu_1 \\ +\rho\frac{\sigma_2}{\sigma_1}\mu_1 - \mu_2 \end{bmatrix}$$

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$\sigma_2^2 + \rho^2 \frac{\sigma_1^2}{\sigma_1^2}$.

$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ 0 & \sigma_2^2 - \rho^2\sigma_2^2 \end{bmatrix}\begin{bmatrix} 1 & -\rho\frac{\sigma_2}{\sigma_1} \\ & 1 \end{bmatrix}$

CHECK VAR-COVAR MATRIX

$$\mathrm{VAR}\begin{pmatrix} W \\ Q \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho\frac{\sigma_2}{\sigma_1} & 1 \end{bmatrix}\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\begin{bmatrix} 1 & -\rho\frac{\sigma_2}{\sigma_1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2(1-\rho^2) \end{pmatrix}$$

3. $\therefore$ INDEPENDENT

$\sigma_2^2\left(1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right) = \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}$.

(b)

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$E\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$VAR\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

Exercise 3

SHOW THAT $\bar{X}$ IND OF $\begin{pmatrix} X_1-\bar{X} \\ \vdots \\ X_n-\bar{X} \end{pmatrix}$ $\quad S^2=\frac{1}{n-1}\sum(X_i-\bar{X})$

$$\begin{pmatrix} \bar{X} \\ X_1-\bar{X} \\ \vdots \\ X_n-\bar{X} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}1' \\ I-\frac{1}{n}11' \end{pmatrix} X = AX.$$

THEN $AX \sim MVN$ BECAUSE $X \sim MVN$.

$$\text{VAR}(\cancel{AX}) = A\left[(\cancel{1}-\rho)I+\rho J\right]A'$$

$$= \begin{pmatrix} \frac{1}{n}1' \\ I-\frac{1}{n}11' \end{pmatrix}\left[(1-\rho)I+\rho J\right]\begin{pmatrix} \frac{1}{n}1 & I-\frac{1}{n}11' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{n} & 0' \\ 0 & \cancel{\quad} \end{pmatrix} \rightarrow \therefore \bar{X}, S^2 \text{ IND.}$$

Exercise 4

$$X_1 = Y_1 Y_3$$

$$X_2 = Y_2 Y_3$$

$$X_3 = Y_3 - Y_1 Y_3 - Y_2 Y_3 = Y_3 (1 - Y_1 - Y_2)$$

JOINT PDF OF X_1, X_2, X_3 IS:

$$f(x_1, x_2, x_3) = f(x_1) \cdot f(x_2) \cdot f(x_3)$$

$$= \frac{x_1^{\alpha_1 - 1} e^{-x_1}}{\Gamma(\alpha_1)} \cdot \frac{x_2^{\alpha_2 - 1} e^{-x_2}}{\Gamma(\alpha_2)} \cdot \frac{x_3^{\alpha_3 - 1} e^{-x_3}}{\Gamma(\alpha_3)}$$

JACOBIAN:

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} \frac{x_2+x_3}{(x_1+x_2+x_3)^2} & -\frac{x_1}{(x_1+x_2+x_3)^2} & -\frac{x_1}{(x_1+x_2+x_3)^2} \\ \frac{\cancel{x_1+x_2}}{(x_1+x_2+x_3)^2} & \frac{\cancel{x_1}+x_3}{(x_1+x_2+x_3)^2} & -\frac{x_2}{(x_1+x_2+x_3)^2} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{y_3^2}.$$

WE CAN ALSO COMPUTE THE JACOBIAN USING

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & 1-y_1-y_2 \end{vmatrix} = y_3^2$$

$$f(y_1,y_2,y_3) = \frac{(y_1y_3)^{\alpha_1-1} e^{-y_1y_3} (y_2y_3)^{\alpha_2-1} e^{-y_2y_3} \left(y_3(1-y_1-y_2)\right)^{\alpha_3-1} e^{-y_3(1-y_1-y_2)}}{\Gamma(\alpha_1)\ \Gamma(\alpha_2)\ \Gamma(\alpha_3)} y_3^2$$

$$= \frac{y_1^{\alpha_1-1} y_2^{\alpha_2-1} (1-y_1-y_2)^{\alpha_3-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \cdot y_3^{\alpha_1+\alpha_2+\alpha_3-1} e^{-y_3} = *$$

NOTE: $\displaystyle\int_0^\infty \frac{y_3^{\alpha_1+\alpha_2+\alpha_3-1} e^{-y_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} dy_3 = 1$

$$\hookrightarrow y_3^{\alpha_1+\alpha_2+\alpha_3-1} e^{-y_3} = \Gamma(\alpha_1+\alpha_2+\alpha_3)$$

$$* = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} y_1^{\alpha_1-1} y_2^{\alpha_2-1} (1-y_1-y_2)^{\alpha_3-1}$$