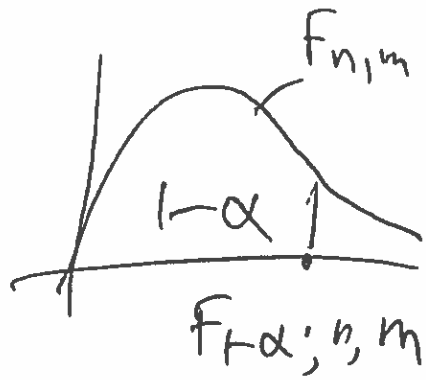
Homework 5 solutions

(a). $X \sim F_{m,n} \rightarrow \frac{1}{X} \sim F_{n,m}$

$$P\left(F_{n,m} < F_{1-\alpha;n,m}\right) = 1-\alpha$$

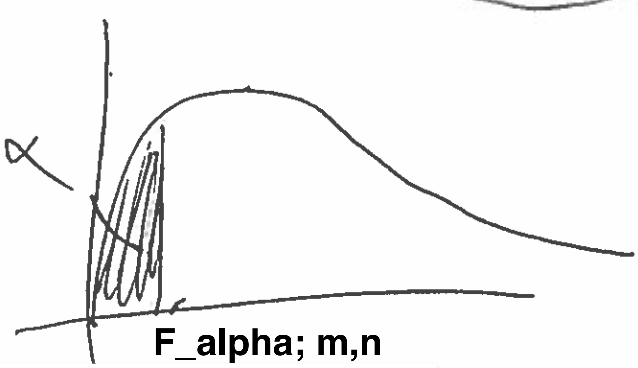


$$P\left(F_{m,n} > \frac{1}{F_{1-\alpha;n,m}}\right) = 1-\alpha$$

$$P\left[F_{m,n} < \frac{1}{F_{1-\alpha;n,m}}\right] = \alpha$$

α

THIS IS THE PERCENTILE of $F_{m,n}$.



F_alpha; m,n

Let X ~ F_1,n, then P(X<F_1-alpha;1,n)=1-a

But X=t_n^2, so P(t_n^2 < F_1-alpha;1,n)=1-alpha

Or P[-sqrt(F_1-alpha;1,n) < t_n < +sqrt(F_1-alpha;1,n)]=1-alpha

Therefore, sqrt(F_1-alpha;1,n)=t_1-alpha/2;n

Finally, t^2_1-alpha/2;n = F_1-alpha;1,n (b). $\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16}}\right)$

OR $\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \sigma_1\sqrt{\frac{1}{13} + \frac{5}{16}}\right)$

AND $\frac{(13-1)S_x^2}{\sigma_1^2} + \frac{(16-1)S_y^2}{\sigma_2^2} \sim \chi^2_{27}$

OR $\frac{12 S_x^2}{\sigma_1^2} + \frac{15 S_y^2}{5\sigma_1^2} \sim \chi^2_{27}$

OR $\frac{12 S_x^2}{\sigma_1^2} + \frac{3 S_y^2}{\sigma_1^2} \sim \chi^2_{27}$

THEN

$$\frac{\dfrac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sigma_1\sqrt{\frac{1}{13} + \frac{5}{16}}}}{\sqrt{\dfrac{\frac{12 S_x^2 + 3 S_y^2}{\sigma_1^2}}{27}}} \sim t_{27}$$

1(c). $\sum X_i \sim \Gamma(n\alpha, \beta) \to M_{\sum X_i}(t) = (1-\beta t)^{-n\alpha}$

$$M_{\bar{X}}(t) = M_{\frac{\sum X_i}{n}}(t) = M_{\sum X_i}\left(\frac{t}{n}\right)$$

$$= \left(1 - \frac{\beta t}{n}\right)^{-n\alpha}$$

LET $Q = \frac{2n}{\beta}\bar{X}$

THEN

$$M_Q(t) = (1-2t)^{-n\alpha}$$

OR $M_Q(t) = (1-2t)^{-\frac{2n\alpha}{2}}$

$\therefore Q = \frac{2n\bar{X}}{\beta} \sim \chi^2_{2n\alpha}$

(ASSUME α IS INTEGER).

(f). $X \sim F_{n,m}$

OR $\dfrac{\chi^2_n/n}{\chi^2_m/m} =$

$$EX = \left(E\frac{\chi^2_n}{n}\right) E\left(\frac{\chi^2_m}{m}\right)^{-1}$$

BECAUSE χ^2_n, χ^2_m ARE IND.

BUT $\chi^2_m \sim \Gamma\left(\frac{m}{2}, 2\right)$

AND $E\dfrac{\chi^2_n}{n} = 1$

SO, $EX = E\left(\Gamma\left(\frac{m}{2}, 2\right)\right)^{-1}$

USE PROPERTY OF $\Gamma(\alpha, \beta)$:

$$EX^k = \frac{\Gamma(\alpha+k)\,\beta^k}{\Gamma(\alpha)}$$

HERE $k = -1$.

(e). $X \sim t_n$

$$X = \frac{Z}{\sqrt{U/n}} \qquad Z \sim N(0,1) \quad U \sim \chi^2_n \qquad Z, U \text{ ARE INDEPENDENT}$$

$$EX = \sqrt{n}\, E Z\, E U^{-1/2} = 0 \quad \text{BECAUSE } EZ = 0$$

$$\text{VAR}(X) = n\left[E Z^2\right]\left[E U^{-1}\right] = \cancel{x}$$

BUT $U \sim \chi^2_n$ OR $U \sim \Gamma\left(\frac{n}{2}, 2\right)$

$$= n\left(\sigma^2 + \mu^2\right) \frac{\Gamma\left(\frac{n}{2} - 1\right) 2^{-1}}{\Gamma\left(\frac{n}{2}\right)}$$

$$= n(1+0) \frac{\Gamma\left(\frac{n}{2} - 1\right)}{\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{n}{2} - 1\right)} \cdot 2^{-1}$$

$$= \frac{n}{n-2}.$$

(d)

$$Y'\Sigma^{-1}Y = (Y_1\ \ Y_2)\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{21} & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}$$

$$= (Y_1\ \ Y_2)\begin{pmatrix}\dfrac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & -\dfrac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\\[2ex] \dfrac{-\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & \dfrac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\end{pmatrix}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}$$

$$= \left(\frac{Y_1\sigma_2^2 - Y_2\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2},\ \frac{-Y_1\sigma_{12} + Y_2\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\right)\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}$$

$$= \frac{Y_1^2\sigma_2^2 - Y_1Y_2\sigma_{12} - Y_1Y_2\sigma_{12} + Y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}$$

$$= \frac{Y_1^2\sigma_2^2 - 2Y_1Y_2\rho\sigma_1\sigma_2 + Y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2} = \frac{Y_1^2\sigma_2^2 - 2Y_1Y_2\rho\sigma_1\sigma_2 + Y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

Now: $$Y'\Sigma^{-1}Y - \frac{Y_1^2}{\sigma_1^2} = \frac{Y_1^2\sigma_2^2 - 2Y_1Y_2\rho\sigma_1\sigma_2 + Y_2^2\sigma_1^2 - \sigma_2^2(1-\rho^2)Y_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

$$= \frac{\cancel{Y_1^2\sigma_2^2} - 2Y_1Y_2\rho\sigma_1\sigma_2 + Y_2^2\sigma_1^2 - \cancel{Y_1^2\sigma_2^2} + \rho^2Y_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

$$= \frac{1}{1-\rho^2}\left[\frac{\rho^2Y_1^2}{\sigma_1^2} - \frac{2Y_1Y_2\rho}{\sigma_1\sigma_2} + \frac{Y_2^2}{\sigma_2^2}\right] = \frac{1}{1-\rho^2}\left(\frac{\rho Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}\right)^2$$

$$= \frac{Q^2}{1-\rho^2}, \quad \text{where } Q \sim N\left(0, \sqrt{1-\rho^2}\right)$$

$$\therefore \sim \chi^2_1$$

(g). FROM CLASS NOTES

$$(\underline{Y}-\underline{\mu})' \underline{\Sigma}^{-1} (\underline{Y}-\underline{\mu}) \sim \chi^2_n \qquad (1)$$

$(X_1, Y_1), \ldots, (X_n, Y_n)$ IS A RANDOM SAMPLE FROM $N_n(\underline{\mu}, \Sigma)$

THEREFORE

$$M_{\bar{X},\bar{Y}}(t_1, t_2) = E e^{t_1 \bar{X} + t_2 \bar{Y}}$$

$$= E e^{t_1 \left(\frac{X_1 + \cdots + X_n}{n}\right) + t_2 \left(\frac{Y_1 + \cdots + Y_n}{n}\right)}$$

$$= \left\{ E e^{\frac{t_1}{n} X_1 + \frac{t_2}{n} Y_1} \right\} \times \cdots \times \left\{ E e^{\frac{t_1}{n} X_n + \frac{t_2}{n} Y_n} \right\}$$

$$= \left\{ M_{X_i, Y_i}\left(\frac{t_1}{n}, \frac{t_2}{n}\right) \right\}^n$$

$$= \left\{ e^{\frac{\underline{t}'}{n}\underline{\mu} + \frac{1}{2} \frac{\underline{t}'}{n} \Sigma \frac{\underline{t}}{n}} \right\}^n = e^{\underline{t}'\underline{\mu} + \frac{1}{2} \underline{t}' \frac{\Sigma}{n} \underline{t}}$$

$$\therefore \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2\left(\underline{\mu}, \frac{\Sigma}{n}\right) \qquad \text{USING } (1)$$

WE GET $(\bar{X}-\mu_1, \bar{Y}-\mu_2) \left(\frac{\Sigma}{n}\right)^{-1} \begin{bmatrix} \bar{X}-\mu_1 \\ \bar{Y}-\mu_2 \end{bmatrix} \sim \chi^2_2$.

(h) $Z \sim N(0,1)$ $\quad$ $U \sim \chi^2_n$

Z AND U ARE INDEPENDENT.

$$f(z,u) = f(z) \cdot f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{u^{\frac{n}{2}-1} e^{-u/2}}{\Gamma(\frac{n}{2}) 2^{n/2}}$$

$$\left.\begin{aligned} x &= \frac{z}{\sqrt{u/n}} \\ w &= u \end{aligned}\right\} \begin{aligned} &\rightarrow z = x\sqrt{\tfrac{w}{n}} \\ &\rightarrow u = w \end{aligned}$$

$$J = \begin{vmatrix} \frac{dx}{dz} & \frac{dx}{dw} \\ \frac{dw}{dz} & \frac{dw}{du} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{u/n}} & -\frac{1}{2}\sqrt{n}\, z\, u^{-3/2} \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{w/n}}$$

THEREFORE,

$$f(x,w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 \frac{w}{n}} \frac{w^{\frac{n}{2}-1} e^{-w/2}}{\Gamma(\frac{n}{2}) 2^{n/2}} \left(\frac{w}{n}\right)^{1/2}$$

NOW INTEGRATE THE JOINT PDF OF X, W W.R.T. W TO FIND $f(x)$:

$$f(x) = \int_0^\infty f(x,w)\,dw$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma\left(\frac{n}{2}\right)\,2^{n/2}\sqrt{n}} \int_0^\infty \underbrace{w^{\frac{n+1}{2}-1}\, e^{-w\Big/\frac{1}{\left(\frac{1}{2}\frac{x^2}{n}+\frac{1}{2}\right)}}}_{\text{KERNEL FUNCTION OF } \Gamma\left(\frac{n+1}{2},\ \frac{2}{\frac{x^2}{n}+1}\right)}\, dw$$

$$= \frac{\left(\frac{2}{\frac{x^2}{n}+1}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\frac{n}{2}\right)\,2^{n/2}\sqrt{n}} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

(i). $Z \sim N(\delta, 1)$

$U \sim \chi^2_n$

THEN $t = \dfrac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$ $(NCP = \delta)$

$$E\,t = \sqrt{n}\,(E\,Z)\left(E\,U^{-1/2}\right)$$

$$= \sqrt{n} \cdot \delta \cdot \frac{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right) 2^{-1/2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\mathrm{VAR}(t) = E\,t^2 - (E\,t)^2$$

(j). $U \sim \chi^2_{n_1}$ (ncp $= \theta$)

$V \sim \chi^2_{n_2}$

$$\frac{U/n_1}{V/n_2} \sim F_{n_1, n_2} \text{ (ncp } = \theta)$$

$$E\, \frac{U/n_1}{V/n_2} = \frac{n_2}{n_1} \left(E\, U\right)\left(E\, V^{-1}\right)$$

NOTE: $M_U(t) = (1-2t)^{-\frac{n_1}{2}} e^{\theta \frac{t}{1-2t}}$