HW 7 100B - SOLUTIONS

EXERCISE 1:

$$f(y) = \frac{2y}{\theta} e^{-\frac{y^2}{\theta}},$$

Find pdf of Y^2 using the method of cdf.

$$F_X(x) = P(X \le x) = P(Y^2 \le x)$$

Let $X = Y^2$

$$= P\underset{y>0}{(} Y \le \sqrt{x}) = F_Y(\sqrt{x})$$

$$\therefore f_X(x) = \frac{1}{2} x^{-1/2} f_Y(\sqrt{x}) = \frac{1}{2} x^{-1/2} \frac{2x^{1/2}}{\theta} e^{-\frac{x}{\theta}} = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

THEREFORE $Y^2 \sim \exp\left(\frac{1}{\theta}\right) \rightarrow EY^2 = \theta$

$$\hat{\theta} = \frac{\sum Y_i^2}{n}$$

$$E\hat{\theta} = \frac{\sum EY_i^2}{n} = \frac{n\theta}{n} = \theta.$$

EXERCISE 2:

$$E(T-\theta)^2 = MSE(T) = VAR(T) + B^2$$

$$ET = \alpha_1\theta + \alpha_2\theta$$

$$B = ET - \theta = (\alpha_1 + \alpha_2 - 1)\,\theta$$

$$VAR(T) = VAR(\alpha_1 \bar{X} + \alpha_2\, cS)$$

$$= \alpha_1^2\, VAR(\bar{X}) + \alpha_2^2\, VAR(cS)$$

$$= \alpha_1^2 \frac{\sigma^2}{n} + \alpha_2^2 \left[E(cS)^2 - (E\,cS)^2\right]$$

$$= \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 \left[c^2\theta^2 - \theta^2\right]$$

$$= \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 (c^2-1)\theta^2$$

MINIMIZE $MSE(T)$ W.R.T. α_1 AND α_2

$$\text{MIN } Q = \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 (c^2-1)\theta + (\alpha_1+\alpha_2-1)^2\theta^2$$

$$\frac{\partial Q}{\partial \alpha_1} = 0 \quad \text{AND} \quad \frac{\partial Q}{\partial \alpha_2} = 0$$

TO GET $\alpha_1 = \dfrac{n(c^2-1)}{c^2 + n(c^2-1)}$ AND $\alpha_2 = \dfrac{1}{c^2 + n(c^2-1)}$

EXERCISE 3:

(a). $$r = \left\{ (X_1-\mu_1)^2 + \cdots + (X_n-\mu_n)^2 \right\}^{1/2}$$

$$r = \sigma \left\{ \underbrace{\left(\frac{X_1-\mu_1}{\sigma}\right)^2 + \cdots + \left(\frac{X_n-\mu_n}{\sigma}\right)^2}_{\chi^2_n \to \Gamma\left(\frac{n}{2},\, 2\right)} \right\}^{1/2}$$

$$Er = \sigma\, E\left(\chi^2_n\right)^{1/2}$$

$$= \sigma \, \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \, 2^{1/2}$$

(b). $S_p^2 = \dfrac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2}$

$\dfrac{(n+m-2)S_p^2}{\sigma^2} = \dfrac{(n-1)S_1^2}{\sigma^2} + \dfrac{(m-1)S_2^2}{\sigma^2}$

$\chi^2_{n+m-2} \qquad \chi^2_{n-1} \qquad \chi^2_{m-1}$

$\dfrac{(n+m-2)S_p^2}{\sigma^2} \sim \chi^2_{n+m-2}$

LET $Q = \dfrac{(n+m-2)S_p^2}{\sigma^2}$ THEN $M_Q(t) = (1-2t)^{-\frac{n+m-2}{2}}$

$S_p^2 = \dfrac{\sigma^2}{n+m-2} Q$ AND $M_{S_p^2}(t) = M_Q\left(\dfrac{\sigma^2}{n+m-2} t\right)$

$= \left(1 - \dfrac{2\sigma^2}{n+m-2} t\right)^{-\frac{n+m-2}{2}}$ $\therefore S_p^2 \sim \Gamma\left(\dfrac{n+m-2}{2}, \dfrac{2\sigma^2}{n+m-2}\right)$

$E S_p = E(S_p^2)^{1/2} = \dfrac{\Gamma\left(\frac{n+m-2}{2} + \frac{1}{2}\right)\left(\frac{2\sigma^2}{n+m-2}\right)^{1/2}}{\Gamma\left(\frac{n+m-2}{2}\right)}$

$= \sigma \sqrt{\dfrac{2}{n+m-2}} \dfrac{\Gamma\left(\frac{n+m-1}{2}\right)}{\Gamma\left(\frac{n+m-2}{2}\right)}$

EXERCISE 4:

(a). $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} X_i^2}$ $(\mu = 0)$

$f(x) = (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2} X_i^2}$

$\ln f(x) = -\frac{1}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} X_i^2$

FIND CRAMÉR-RAO LOWER BOUND:

$$\frac{d \ln f(x)}{d\sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} X_i^2$$

$$\frac{d^2 \ln f(x)}{d\sigma^2 (2)} = \frac{1}{2\sigma^4} - \frac{1}{\sigma^6} X_i^2$$

$$I(\theta) = -E\left(\frac{1}{2\sigma^4} - \frac{X_i^2}{\sigma^6}\right) = -\left(\frac{1}{2\sigma^4} - \frac{\sigma^2}{\sigma^6}\right) = +\frac{1}{2\sigma^4}$$

CRAMÉR-LOWER BOUND: $\frac{1}{n I(\theta)} = \frac{2\sigma^4}{n}$

OUR ESTIMATOR HAS: $E\hat{\sigma}^2 = E\frac{\sum X_i^2}{n} = \frac{n\sigma^2}{n} = \sigma^2$ UNBIASED

AND $VAR\left(\frac{\sum X_i^2}{n}\right) = \frac{\sigma^4}{n^2} VAR\left(\frac{\sum X_i^2}{\sigma^2}\right) = \frac{2n\sigma^4}{n^2} = \frac{2\sigma^4}{n}$ YES IT'S EFFICIENT.

(b). $X_1, \ldots, X_m$ iid $\exp(\lambda_1)$

$$\sum X_i \sim \Gamma\left(m, \frac{1}{\lambda_1}\right)$$

SIMILARLY, $\sum Y_i \sim \Gamma\left(n, \frac{1}{\lambda_2}\right)$

USE MGF TO SHOW IT:

$$M_{\sum X_i}(t) = \left(M_{X_i}(t)\right)^m = \left(1 - \frac{t}{\lambda_1}\right)^{-m}$$

BUT THIS IS $\Gamma\left(m, \frac{1}{\lambda_1}\right)$

THEREFORE $2\lambda_1 \sum X_i \sim \chi^2_{2m}$ AND $2\lambda_2 \sum Y_i \sim \chi^2_{2n}$

$$\frac{2\lambda_1 \sum X_i}{2\lambda_2 \sum Y_i} = \frac{\lambda_1}{\lambda_2} \frac{\bar{X}}{\bar{Y}} \sim F_{2m, 2n}$$

$$M_{2\lambda_1 \sum X_i}(t) = M_{\sum X_i}(2\lambda_1 t)$$

$$= \left(1 - \frac{2\lambda_1 t}{\lambda_1}\right)^{-\frac{2m}{2}} = (1 - 2t)^{-\frac{2m}{2}}$$

WHICH IS χ^2_{2m}

EXERCISE 5

$$L = (2\pi\sigma^2)^{-n/2} |\underline{V}|^{-1/2} e^{-\frac{1}{2\sigma^2}(\underline{y}-\mu\underline{1})'\underline{V}^{-1}(\underline{y}-\mu\underline{1})}$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2}\ln|V| - \frac{1}{2\sigma^2}(y-\mu\underline{1})' V^{-1}(y-\mu\underline{1})$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2}\ln|\underline{V}| - \frac{1}{2\sigma^2}\left(\underline{y}'\underline{V}^{-1}\underline{y} - 2\mu\underline{1}'\underline{V}^{-1}\underline{y} + \mu^2\underline{1}'\underline{V}^{-1}\underline{1}\right)$$

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{2\sigma^2}\left(-2\underline{1}'\underline{V}^{-1}\underline{y} + 2\mu\underline{1}'\underline{V}^{-1}\underline{1}\right) = 0 \rightarrow \boxed{\hat{\mu} = \frac{\underline{1}'\underline{V}^{-1}\underline{y}}{\underline{1}'\underline{V}^{-1}\underline{1}}}$$

AND $$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(\underline{y}-\mu\underline{1})'\underline{V}^{-1}(\underline{y}-\mu\underline{1}) = 0$$

$$\hat{\sigma}^2 = \frac{(\underline{y}-\hat{\mu}\underline{1})'\underline{V}^{-1}(\underline{y}-\hat{\mu}\underline{1})}{n}$$

<u>NOTE</u>: $\left.\begin{array}{l} E\underline{y} = \mu\underline{1} \\ VAR(\underline{y}) = \sigma^2\underline{V} \end{array}\right\}$

[redacted]:

$$E\hat{\mu} = E\,\frac{1'V^{-1}y}{1'V^{-1}1} = \frac{1'V^{-1}\mu 1}{1'V^{-1}1} = \mu.$$

$$E\hat{\sigma}^2 = \frac{1}{n} E\, tr\,(y-\hat{\mu}1)'V^{-1}(y-\hat{\mu}1)$$

$$= \frac{1}{n}\, tr\, V^{-1}\left\{E\,(y-\hat{\mu}1)(y-\hat{\mu}1)'\right\}$$

$$= \frac{1}{n}\, tr\, V^{-1}\left\{VAR\,(y-\hat{\mu}1) + \underbrace{E(y-\hat{\mu}1)}_{0}\cdot\underbrace{(E\,y-\hat{\mu}1)'}_{0}\right\}$$

NEED $VAR\,(y-\hat{\mu}1) = VAR\left(y - \frac{11'V^{-1}y}{1'V^{-1}1}\right) = VAR\left(I - \frac{11'V^{-1}}{1'V^{-1}1}\right)y$

$$= \sigma^2\left(I - \frac{11'V^{-1}}{1'V^{-1}1}\right)V\left(I - \frac{V^{-1}11'}{1'V^{-1}1}\right) = \sigma^2\left(V - \frac{11'}{1'V^{-1}1}\right)$$

$$= \frac{1}{n}\, tr\, V^{-1}\left(\sigma^2\left(V - \frac{11'}{1'V^{-1}1}\right)\right)$$

$$= \frac{1}{n}\sigma^2\left(tr\, I - tr\,\frac{V^{-1}11'}{1'V^{-1}1}\right) = \frac{1}{n}\sigma^2\left(n - tr\,\frac{1'V^{-1}1}{1'V^{-1}1}\right)$$

$$= \frac{n-1}{n}\sigma^2.$$

$E\hat{\mu} = \mu$ AND $VAR(\hat{\mu}) = VAR\left(\frac{\underline{1}'V^{-1}\underline{y}}{\underline{1}'V^{-1}\underline{1}}\right)$

$$= \sigma^2 \frac{\underline{1}'V^{-1}V\,V^{-1}\underline{1}}{(\underline{1}'V^{-1}\underline{1})^2} = \frac{\sigma^2}{\underline{1}'V^{-1}\underline{1}}$$

$$I(\underline{\theta}) = -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial\mu^2} & \frac{\partial^2 \ln L}{\partial\mu\,\partial\sigma^2} \\ \frac{\partial^2 \ln L}{\partial\sigma^2\,\partial\mu} & \frac{\partial^2 \ln L}{\partial(\sigma^2)^2} \end{pmatrix}$$

$$\frac{\partial^2 \ln L}{\partial\mu^2} = -\frac{\underline{1}'V^{-1}\underline{1}}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial\mu\,\partial\sigma^2} = \frac{1}{2\sigma^4}\left(-2\,\underline{1}'V^{-1}\underline{y} + 2\mu\,\underline{1}'V^{-1}\underline{1}\right)$$

$$\frac{\partial^2 \ln L}{\partial(\sigma^2)^2} = -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}(\underline{y}-\mu\underline{1})'V^{-1}(\underline{y}-\mu\underline{1})$$

FINALLY AFTER TAKING EXPECTATION WE GET

$$I(\theta)^{-1} = \begin{pmatrix} \frac{\sigma^2}{\underline{1}'V^{-1}\underline{1}} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

$\therefore \hat{\mu}$ IS EFFICIENT ESTIMATOR OF μ $\quad (\underline{y}-\mu\underline{1})'V^{-1}(\underline{y}-\mu\underline{1}) \sim \chi^2_n$

EXERCISE 6

$$f(x) = \frac{\alpha x^{\alpha-1}}{3^{\alpha}} \qquad 0 \le x \le 3$$

$$EX = \int_0^3 x \, \frac{\alpha x^{\alpha-1}}{3^{\alpha}} \, dx = \frac{\alpha}{3^{\alpha}} \, \frac{x^{\alpha+1}}{\alpha+1} \Big|_0^3 = \frac{3\alpha}{\alpha+1}$$

$$\frac{3\alpha}{\alpha+1} = \bar{X} \quad \rightarrow \quad \hat{\alpha} = \frac{\bar{X}}{3-\bar{X}}.$$

MLE:

$$L = \frac{\alpha^n}{3^{n\alpha}} \left(\prod X_i\right)^{\alpha-1}$$

$$\ln L = n \ln \alpha - n\alpha \ln 3 + (\alpha-1) \ln \prod X_i$$

$$\frac{d \ln L}{d\alpha} = \frac{n}{\alpha} - n \ln 3 + \ln \prod X_i = 0$$

$$\hat{\alpha} = \frac{n}{n \ln 3 - \ln \prod X_i}.$$

EXERCISE 7

FIRST FIND THE JOINT PDF OF $X_{(1)}, X_{(n)}$

(SEE HANDOUT #22, PAGE 1, (e)
WITH $i=1$, $j=n$)

HERE: $f(x) = \frac{1}{\alpha}$, $F(x) = \frac{x}{\alpha}$

$$\therefore\ g_{X_{(1)}X_{(n)}}(u,v) = \frac{n!}{(1-1)!\,(n-1-1)!\,(n-n)!}\,\frac{1}{\alpha}\cdot\frac{1}{\alpha}\left(\frac{u}{\alpha}\right)^{1-1}\left[\frac{v}{\alpha}-\frac{u}{\alpha}\right]^{n-1-1}\left[1-\frac{v}{\alpha}\right]^{n-n}$$

$$g_{X_{(1)}X_{(n)}}(u,v) = n(n-1)\,\frac{1}{\alpha^2}\,\frac{1}{\alpha^{n-2}}\,[v-u]^{n-2}$$

$$= \frac{n(n-1)\,(v-u)^{n-2}}{\alpha^n}$$

NOW WE NEED THE JACOBIAN:

$$\left.\begin{aligned} R &= X_{(n)} - X_{(1)} \\ Q &= \frac{X_{(1)} + X_{(n)}}{2} \end{aligned}\right\} \qquad \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}+\frac{1}{2} = 1$$

SOLVE FOR $X_{(1)}$ AND $X_{(n)}$ TO GET $X_{(1)} = Q - \frac{R}{2}$, $X_{(n)} = Q + \frac{R}{2}$

FINALLY $f_{R,Q}(r,q) = \frac{n(n-1)}{\alpha^n}\, r^{n-2}$