EXERCISE 1

Answer the following questions:

a. The lifetime of certain batteries are supposed to have a variance of 150 hours$^2$. Using $\alpha = 0.05$ test the following hypothesis

$H_0 : \sigma^2 = 150$

$H_a : \sigma^2 > 150$

if the lifetimes of 15 of these batteries (which constitutes a random sample from a normal population) have:

\[
\sum_{i=1}^{15} x_i = 250, \quad \sum_{i=1}^{15} x_i^2 = 8000.
\]

where $X$ denotes the lifetime of a battery.

b. A confidence interval is unbiased if the expected value of the interval midpoint is equal to the estimated parameter. For example the midpoint of the interval $\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ is $\bar{x}$, and $E(\bar{x}) = \mu$.

Now consider the confidence interval for $\sigma^2$. Show that the expected value of the midpoint of this confidence interval is not equal to $\sigma^2$.

EXERCISE 2

Let $X$ be a uniform random variable on $(0, \theta)$. You have exactly one observation from this distribution and you want to test the null hypothesis $H_0 : \theta = 10$ against the alternative $H_a : \theta > 10$, and you want to use significance level $\alpha = 0.10$. Two testing procedures are being considered:

Procedure $G$ rejects $H_0$ if and only if $X \geq 9$.

Procedure $K$ rejects $H_0$ if either $X \geq 9.5$ or if $X \leq 0.5$.

a. Confirm that Procedure $G$ has a Type I error probability of 0.10.

b. Confirm that Procedure $K$ has a Type I error probability of 0.10.

c. Find the power of Procedure $G$ when $\theta = 12$.

d. Find the power of Procedure $K$ when $\theta = 12$.

EXERCISE 3

Consider the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, \ldots n$. Find a confidence interval for $\sigma^2$. *Hint:* In your class notes you can find the distribution of $S^2$ which can be used as the pivotal quantity to construct the confidence interval. Use $1 - \alpha$ confidence level.
EXERCISE 4
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter $\lambda$. Find the best critical region for testing
   
   $H_0 : \lambda = 2$
   
   $H_a : \lambda = 5$
   
   using the Neyman-Pearson lemma.

b. Let $Y_1, Y_2, \ldots, Y_n$ be the outcomes of $n$ independent Bernoulli trials. Find the best critical region for testing
   
   $H_0 : p = p_0$
   
   $H_a : p > p_0$
   
   using the Neyman-Pearson lemma.

EXERCISE 5
Suppose that the length in millimeters of metal fibers produced by a certain process follow the normal distribution with mean $\mu$ and standard deviation $\sigma$ (both are unknown). We will test:

$H_0 : \mu = 5.2$

$H_a : \mu \neq 5.2$

A sample size of $n = 15$ metal fibers was selected and was found that $\bar{x} = 5.4$ and $s = 0.4266$.

a. Approximate the $p$-value using only your $t$ table and use it to test this hypothesis. Assume $\alpha = 0.05$.

b. Assume now that the population standard deviation is known and it is equal to $\sigma = 0.4266$. Compute the power of the test when the actual mean is $\mu_a = 5.35$ and you can accept $\alpha = 0.05$.

c. On the previous page draw the two distributions (under $H_0$ and under $H_a$) and show the Type I error and the Type II error on them.

d. Assume now that the hypothesis we are testing is
   
   $H_0 : \mu = 5.2$
   
   $H_a : \mu > 5.2$
   
   Determine the sample size needed in order to detect with probability 95% a shift from $\mu_0 = 5.2$ to $\mu_a = 5.3$ if you are willing to accept a Type I error $\alpha = 0.05$. Assume $\sigma = 0.4266$. 