Exercise 1
The confidence interval for the ratio of two normal population variances \( \frac{s_1^2}{s_2^2} \) is:

\[
\frac{s_1^2}{s_2^2} \frac{1}{F_{1-n_1-1,n_2-2}} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{s_1^2}{s_2^2} \frac{1}{F_{\frac{n_2-1}{n_1-1}}}
\]

In the above confidence interval, \( s_1^2 \) and \( s_2^2 \) are the sample variances based on two independent samples of size \( n_1, n_2 \) selected from two normal populations \( N(\mu_1, \sigma_1) \) and \( N(\mu_2, \sigma_2) \). Here we use the result

\[
\frac{s_1^2}{s_2^2} \sim F_{n_1-1, n_2-1}.
\]

Exercise 2
We start by finding the distribution of \( X_p - \tilde{X} \).
\( E(X_p - \tilde{X}) = 0 \) and \( \text{Var}(X_p - \tilde{X}) = \sigma^2(1 + \frac{1}{n}) \). The distribution of \( X_p - \tilde{X} \) is:

\[
X_p - \tilde{X} \sim N(0, \sigma \sqrt{\frac{1}{1 + \frac{1}{n}}})
\]

\[
Z = \frac{X_p - \tilde{X}}{\sigma \sqrt{1 + \frac{1}{n}}}
\]

\[
t = \frac{\sigma \sqrt{1 + \frac{1}{n}}}{s \sqrt{\frac{n - 1}{n}}} \Rightarrow t = \frac{X_p - \tilde{X}}{s \sqrt{1 + \frac{1}{n}}}.
\]

Since the above ratio follows the \( t \) distribution with \( n - 1 \) degrees of freedom the \( 1 - \alpha \) prediction interval for \( X_p \) is:

\[
P(-t_{\alpha/2, n-1} \leq \frac{X_p - \tilde{X}}{s \sqrt{1 + \frac{1}{n}}} \leq t_{\alpha/2, n-1}) = 1 - \alpha
\]

Or \( X_p \) will fall in \( \tilde{X} \pm t_{\alpha/2, n-1} s \sqrt{1 + \frac{1}{n}} \).

Exercise 3
We found earlier that \( nI(\theta) = \frac{2}{\theta^2} \), and the lower bound of the Cramér-Rao inequality is \( \frac{4}{n^2} \). Using the asymptotic properties of the maximum likelihood estimates the 95% confidence interval for \( \lambda \) is:

\[
\tilde{X} \pm Z \sqrt{\frac{\lambda}{n}} \text{ or } \tilde{X} \pm Z \sqrt{\frac{\lambda}{n}}
\]

We replace \( \lambda \) with its mle estimate, \( \hat{\lambda} = \tilde{X} \). From the data we compute \( \tilde{x} = 24.9 \), therefore the 95% confidence interval is:

\[
24.9 \pm 1.96 \sqrt{\frac{24.9}{23}} \text{ or } 24.9 \pm 2.04
\]

or 22.86 \( \leq \lambda \leq 26.94 \).

Exercise 4
\( \tilde{X} - \tilde{Y} \sim N(\mu_1 - \mu_2, \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}) \). Therefore we can write:

\[
Z = \frac{\tilde{X} - \tilde{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}
\]

And since \( s_1^2 = 3s_2^2 \) we get:

\[
Z = \frac{\tilde{X} - \tilde{Y} - (\mu_1 - \mu_2)}{\sqrt{s_2^2 \left( \frac{1}{3} + \frac{1}{n_2} \right)}}
\]

Now we need to define a \( \chi^2 \) random variable. Because \( X \) and \( Y \) are independent we have:

\[
\frac{(9 - 1)S_1^2}{\sigma_1^2} + \frac{(12 - 1)S_2^2}{\sigma_2^2} \sim \chi^2_{12 + 9 - 2} \sim \chi^2_{19}.
\]
Using again $\sigma_1^2 = 3\sigma_2^2$ we get:

$$\frac{4.8S_1^2 + 11S_2^2}{\sigma_2^2} \sim \chi^2_{19}. $$

Now we can define a variable that has a $t$ distribution as follows:

$$ t = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{4.8S_1^2 + 11S_2^2}{\sigma_2^2} + \frac{1}{19}}} \sim t_{19} $$

Finally we get:

$$ t = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{4.8S_1^2 + 11S_2^2}{\sigma_2^2} + \frac{1}{19}}} \sim t_{19} \sqrt{\frac{228}{5}}. $$

We can use the above $t_{19}$ random variable to construct a 95% confidence interval for $\mu_1 - \mu_2$. We want:

$$ P(-t_{\frac{0.05}{2},19} \leq \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{4.8S_1^2 + 11S_2^2}{\sigma_2^2} + \frac{1}{19}}} \leq t_{\frac{0.95}{2},19}) = 1 - \alpha. $$

After some manipulation we find that $\mu_1 - \mu_2$ will fall in the following interval with 95% confidence:

$$ \bar{X} - \bar{Y} \pm t_{\frac{0.05}{2},19} \sqrt{\frac{5}{3} S_1^2 + 11S_2^2} \sqrt{\frac{228}{5}}. $$
Exercise 5

First find the joint PDF of $X(1)$ and $X(n)$:

\[ f(x) = \frac{1}{\alpha} \quad F(x) = \frac{x}{\alpha} \]

\[ g_{X(1), X(n)}(u,v) = \binom{n}{1-1} \binom{n-1}{n-1-1} \frac{1}{\alpha^2} \left( \frac{v}{\alpha} \right) \left( \frac{v-u}{\alpha} \right) \left( 1 - \frac{v}{\alpha} \right) \]

\[ g_{X(1), X(n)}(u,v) = \frac{n(n-1)(v-u)^{n-2}}{\alpha^n} \]

Now we need the Jacobian:

\[ R = X(n) - X(1) \]

\[ Q = \frac{X(1) + X(n)}{2} \]

\[ \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = 1 \]

Solve for $X(1)$ and $X(n)$ in terms of $Q$ and $R$:

\[ X(1) = Q - \frac{R}{2} \quad \text{and} \quad X(n) = Q + \frac{R}{2} \]

Finally,

\[ f_{R,Q}(r,y) = \frac{n(n-1)r^{n-2}}{\alpha^n} \]
PDF of $j$th order statistic:

$$g_{X(j)}(x) = \frac{n!}{(n-j)! \, (j-1)!} x^{j-1} (1-x)^{n-j-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-j+1) \, \Gamma(j)} 
\times j-1 \, (1-x)^{(n-j+1)-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-j+1) \, \Gamma(j)} 
\times j-1 \, (1-x) \left\{ \frac{(n-j+1)-1}{B(j, n-j+1)} \right\}$$

where, $B(j, n-j+1) = \frac{\Gamma(j) \, \Gamma(n-j+1)}{\Gamma(n+1)}$