EXERCISES 1
Find the distribution of the random variable $X$ for each of the following moment-generating functions:

a. $M_X(t) = \left(\frac{1}{3}e^t + \frac{2}{3}\right)^5$. $X$ follows binomial with $n = 5$, $p = \frac{1}{3}$.

b. $M_X(t) = \frac{e^t}{2e^t} - \frac{1}{2}t + \frac{1}{2}e^{-t}$. $X$ follows geometric with $p = \frac{1}{2}$.

c. $M_X(t) = e^{2(e^t - 1)}$. $X$ follows Poisson with $\lambda = 2$.

EXERCISES 2
Let $M_X(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$ be the moment-generating function of a random variable $X$.

a. To find $E(X)$ we need the first derivative of $M_X(t)$ w.r.t. $t$:

$$M_X'(t) = \frac{1}{6}e^t + \frac{2}{6}2e^{2t} + \frac{3}{6}3e^{3t}.$$ The first moment is the previous derivative evaluated at $t = 0$. The result is $E(X) = \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = \frac{7}{6}$.

b. To find $\text{var}(X)$ we need the second moment. We need the second derivative of $M_X(t)$ w.r.t. $t$.

$$M_X''(t) = \frac{1}{6}e^t + \frac{2}{6}2e^{2t} + \frac{3}{6}3e^{3t}.$$ And at $t = 0$ we obtain the second moment. It is equal to $E(X^2) = \frac{1}{6} + \frac{1}{3} + \frac{3}{2} = 6$. The variance is equal to: $\text{var}(X) = E(X^2) - (E(X))^2 = 6 - \left(\frac{7}{6}\right)^2 = \frac{7}{6}$.

c. From the definition of moment-generating functions $M_X(t) = Ee^{tX}$ we see that $X$ is discrete with possible values $1, 2$, and $3$, and corresponding probabilities $\frac{1}{6}, \frac{1}{3},$ and $\frac{1}{2}$.

EXERCISES 3
Let $X$ follow the Poisson probability distribution with parameter $\lambda$. Its moment-generating function is $M_X(t) = e^{\lambda(e^t - 1)}$.

a. The moment-generating function of $Z = \frac{X - \lambda}{\sqrt{\lambda}}$ is:

$$M_Z(t) = M_{X-\lambda}(t) = e^{-\sqrt{\lambda}t}M_X\left(\frac{t}{\sqrt{\lambda}}\right) = M_Z(t) = e^{-\sqrt{\lambda}t}e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}} - 1\right)}.$$ 

b. Using the series expansion of $e^{\sqrt{\lambda}x} = 1 + \frac{\sqrt{\lambda}x}{1!} + \frac{(\sqrt{\lambda}x)^2}{2!} + \frac{(\sqrt{\lambda}x)^3}{3!} + \cdots$ we get:

$$M_Z(t) = e^{-\sqrt{\lambda}t+\lambda+\frac{t^2}{2\sqrt{\lambda}} + \frac{\lambda^3}{3!(\sqrt{\lambda})^3} + \cdots} = e^{-\sqrt{\lambda}t+\lambda+\frac{t^2}{2\sqrt{\lambda}} + \cdots} = e^{-\sqrt{\lambda}t+\lambda+\frac{t^2}{2\sqrt{\lambda}} + \cdots}.$$ Therefore,

$$\lim_{\lambda \to \infty} M_Z(t) = e^{\frac{t^2}{2}}.$$ In other words, as $\lambda \to \infty$, the ratio $Z = \frac{X - \lambda}{\sqrt{\lambda}}$ converges to the standard normal distribution.

EXERCISES 4
Here we use the normal approximation to Poisson. It is given that $X$ follows the Poisson distribution with $\lambda = 100$. We know that for large $\lambda$ the ratio $Z = \frac{X - \lambda}{\sqrt{\lambda}}$ follows the standard normal distribution. Therefore:

$$P(X \leq 110) = P(Z \leq \frac{110.5 - 100}{\sqrt{100}}) = P(Z < 1.05) = 0.8531.$$ 

The exact probability is

$$P(X \leq 110) = \sum_{x=0}^{110} e^{-100} \frac{100^x}{x!} = \cdots = 0.8529.$$ 

The approximation is not bad!

EXERCISE 5
We know that the moment generating function of a normal random variable is $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. Because the sample is i.i.d the moment generating function of $T = \sum_{i=1}^{n} X_i$ is:

$$M_T(t) = Ee^{tX} = Ee^{X_1 + \cdots + X_n} = Ee^{X_1 t} \cdots Ee^{X_n t} \Rightarrow M_T(t) = e^{n\mu t + \frac{1}{2}n\sigma^2 t^2}.$$ This is the moment generating function of a normal random variable with mean $n\mu$ and variance $n\sigma^2$. Therefore $T \sim N(n\mu, n\sigma^2)$. 

University of California, Los Angeles
Department of Statistics

Statistics 100B
Instructor: Nicolas Christou
EXERCISE 6
The two samples are independent with $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$.

a. $E(X - Y) = E(X) - E(Y) = \mu_1 - \mu_2$.

b. $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$. The covariance is zero because the two samples are independent.

c. $M_{X-Y} = E(e^{(X-Y)t}) = e^{\frac{t}{m}M_{X1} + \frac{t}{n}M_{X2}} = e^{\frac{t}{m}(\mu_1 + \mu_2) + \frac{t}{n}(\frac{1}{m} + \frac{1}{n})}$

$= e^{t(\mu_1 + \mu_2)}$. This is the moment generating function of a normal random variable with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$.

d. It is given that $n = m$ and $\sigma_1^2 = 2, \sigma_2^2 = 2.5$. We want $P(-1 < \bar{X} - \bar{Y} - (\mu_1 - \mu_2) < 1) = 0.95$.

Since $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \sqrt{\frac{2 \sigma_1^2}{m} + \frac{2 \sigma_2^2}{n}})$ we get:

$$P\left(\frac{-1}{\sqrt{\frac{2 \sigma_1^2}{m} + \frac{2 \sigma_2^2}{n}}} < Z < \frac{1}{\sqrt{\frac{2 \sigma_1^2}{m} + \frac{2 \sigma_2^2}{n}}}\right) = 0.95 \Rightarrow P\left(\frac{-1}{\sqrt{\frac{2 \sigma_1^2}{m} + \frac{2 \sigma_2^2}{n}}} < Z < \frac{1}{\sqrt{\frac{2 \sigma_1^2}{m} + \frac{2 \sigma_2^2}{n}}}\right) = 0.95$$

Therefore, $1.96 = \frac{1}{\sqrt{\frac{2 \sigma_1^2}{m} + \frac{2 \sigma_2^2}{n}}} \Rightarrow 1.96 = \frac{1}{\sqrt{\frac{2 \times 2}{m} + \frac{2 \times 2.5}{n}}} \Rightarrow n = 17.3 \approx 18$.

EXERCISE 7
We are given that $X$ follows the normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Therefore its probability density function (pdf) is $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $-\infty < x < \infty$. The random variable $Y$ is said to follow the lognormal distribution if $Y = e^X$ because $\log(y)$ follows the normal distribution. To find its pdf we start with its cumulative distribution function which by definition is:

$$F(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \log(y)) = F_X(\log(y)) \Rightarrow f(y) = \frac{1}{y}f_x(\log(y)) \Rightarrow \frac{1}{y} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\log(y))^2}$$

EXERCISE 8
The radius of a circle, $X$, is an exponential random variable with parameter $\lambda$. Therefore its pdf is $f(x) = \lambda e^{-\lambda x}$. Let $Y$ be the area of the circle. Then $Y = \pi X^2$. We want to find the pdf of $Y$. We start with the c.d.f. function: $F(y) = P(Y \leq y) = P(\pi X^2 \leq y) = P(X^2 \leq \frac{y}{\pi}) = P(-\sqrt{\frac{y}{\pi}} \leq X \leq +\sqrt{\frac{y}{\pi}}) = F_X(\sqrt{\frac{y}{\pi}}) - F_X(-\sqrt{\frac{y}{\pi}}) \Rightarrow f(y) = F(y)' = \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{\pi}} = \lambda e^{-\lambda \sqrt{\frac{y}{\pi}}}$.