EXERCISE 1
Find the distribution of the random variable \( X \) for each of the following moment-generating functions:

a. \( M_X(t) = \left( \frac{1}{3} e^t + \frac{2}{3} \right)^5 \). \( X \) follows binomial with \( n = 5, p = \frac{1}{3} \).

b. \( M_X(t) = \frac{e^t}{2-2e} \). \( X \) follows geometric with \( p = \frac{1}{2} \).

c. \( M_X(t) = e^{2(\varepsilon^t-1)} \). \( X \) follows Poisson with \( \lambda = 2 \).

EXERCISE 2
Let \( M_X(t) = \frac{1}{6} e^t + \frac{2}{6} e^{2t} + \frac{3}{6} e^{3t} \) be the moment-generating function of a random variable \( X \).

a. To find \( E(X) \) we need the first derivative of \( M_X(t) \) w.r.t. \( t \):
\[
M'_X(t) = \frac{1}{6} e^t + \frac{2}{6} 2e^{2t} + \frac{3}{6} 3e^{3t}.
\]
The first moment is the previous derivative evaluated at \( t = 0 \). The result is \( E(X) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = \frac{7}{6} \).

b. To find \( \text{var}(X) \) we need the second moment. We need the second derivative of \( M_X(t) \) w.r.t. \( t \).
\[
M''_X(t) = \frac{1}{6} e^t + \frac{2}{6} 2^2 e^{2t} + \frac{3}{6} 3^2 e^{3t}.
\]
And at \( t = 0 \) we obtain the second moment. It is equal to \( E(X^2) = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = 6 \). The variance is equal to: \( \text{var}(X) = E(X^2) - (E(X))^2 = 6 - \left(\frac{7}{6}\right)^2 = \frac{5}{6} \).

c. From the definition of moment-generating functions \( M_X(t) = Ee^{Xt} \) we see that \( X \) is discrete with possible values 1, 2, and 3, and corresponding probabilities \( \frac{1}{6}, \frac{2}{6}, \) and \( \frac{3}{6} \).

EXERCISE 3
Let \( X \) follow the Poisson probability distribution with parameter \( \lambda \). Its moment-generating function is \( M_X(t) = e^{\lambda(e^t-1)} \).

a. The moment-generating function of \( Z = \frac{X - \lambda}{\sqrt{\lambda}} \) is:
\[
M_Z(t) = M_{X-\lambda}(\frac{t}{\sqrt{\lambda}}) = e^{-\sqrt{\lambda} t} M_X(\frac{t}{\sqrt{\lambda}}) = e^{-\sqrt{\lambda} t} e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}.
\]

b. Using the series expansion of
\[
e^\frac{t}{\sqrt{\lambda}} = 1 + \frac{\frac{t}{\sqrt{\lambda}}}{1!} + \frac{(\sqrt{\lambda})^2}{2!} + \frac{(\sqrt{\lambda})^3}{3!} + \cdots
\]
we get:
\[
M_Z(t) = e^{-\sqrt{\lambda} t} e^{\lambda(1 + \frac{t}{\sqrt{\lambda}} + \frac{(\sqrt{\lambda})^2}{2!} + \frac{(\sqrt{\lambda})^3}{3!} + \cdots)} = e^{-\sqrt{\lambda} t} e^{\lambda(1 + \frac{t^2}{2!} + \frac{\lambda^3}{3!(\sqrt{\lambda})^3} + \cdots)}
\]
Therefore,
\[
\lim_{\lambda \to \infty} M_Z(t) = e^{\frac{t^2}{2}}.
\]
In other words, as \( \lambda \to \infty \), the ratio \( Z = \frac{X - \lambda}{\sqrt{\lambda}} \) converges to the standard normal distribution.

EXERCISE 4
Here we use the normal approximation to Poisson. It is given that \( X \) follows the Poisson distribution with \( \lambda = 100 \). We know that for large \( \lambda \) the ratio \( Z = \frac{X - \lambda}{\sqrt{\lambda}} \) follows the standard normal distribution. Therefore:
\[
P(X \leq 110) = P(Z < \frac{110.5 - 100}{\sqrt{100}}) = P(Z < 1.05) = 0.8531.
\]
The exact probability is
\[
P(X \leq 110) = \sum_{x=0}^{110} e^{-100 \frac{x^2}{2!}} = \cdots = 0.8529.
\]
The approximation is not bad!

EXERCISE 5
We know that the moment generating function of a normal random variable is \( e^{t \mu + \frac{1}{2} \sigma^2 t^2} \). Because the sample is i.i.d the moment generating function of \( T = \sum_{i=1}^{n} X_i \) is:
\[
M_T(t) = Ee^{\lambda t} = Ee^{X_1 t} \cdots Ee^{X_n t} = Ee^{X_1 t} \cdots Ee^{X_n t} \Rightarrow M_T(t) = e^{n \mu t + \frac{1}{2} n \sigma^2 t^2}.
\]
This is the moment generating function of a normal random variable with mean \( n \mu \) and variance \( n \sigma^2 \). Therefore \( T \sim N(n \mu, \sigma \sqrt{n}) \).
EXERCISE 6
The two sample are independent with \( X \sim N(\mu_1, \sigma_1) \) and \( Y \sim N(\mu_2, \sigma_2) \).

a. \( E(X - Y) = E(X) - E(Y) = \mu_1 - \mu_2 \).

b. \( \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} \). The covariance is zero because the two samples are independent.

c. \( M_{X-Y} = E_e^{X-Y, \theta} = E(X^\theta)E(Y^{-\theta}) = M_X(\frac{\theta}{m}) \cdots M_X(\frac{\theta}{m})M_Y(\frac{-\theta}{n}) \cdots M_Y(\frac{-\theta}{n}) = \exp(\frac{\mu_1}{m} + \frac{1}{2}\sigma_1^2 \frac{\theta^2}{m^2} e^{-n(\mu_2 \theta + \frac{1}{2}\sigma_2^2 \frac{\theta^2}{n^2}}) = \exp(\mu_1 - \mu_2 + \frac{1}{2}\sigma_1^2 \frac{\theta^2}{m^2} + \frac{1}{2}\sigma_2^2 \frac{\theta^2}{n^2}) \). This is the moment generating function of a normal random variable with mean \( \mu_1 - \mu_2 \) and variance \( \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n} \).

d. It is given that \( n = m \) and \( \sigma_1^2 = 2, \sigma_2^2 = 2.5 \). We want \( P(-1 < X - \bar{Y} - (\mu_1 - \mu_2) < 1) = 0.95 \).

Since \( X - \bar{Y} \sim N(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}) \) we get:

\[
P\left(\frac{-1}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} < Z < \frac{1}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}\right) = 0.95 \Rightarrow P\left(\frac{-1}{\sqrt{\frac{\sigma_1^2}{n}} + \frac{\sigma_2^2}{n}} < Z < \frac{1}{\sqrt{\frac{\sigma_1^2}{n}} + \frac{\sigma_2^2}{n}}\right) = 0.95
\]

Therefore, \( 1.96 = \frac{1}{\sqrt{\frac{\sigma_1^2}{n}} + \frac{\sigma_2^2}{n}} \Rightarrow 1.96 = \frac{1}{\sqrt{\frac{\sigma_1^2}{n}} + \frac{\sigma_2^2}{n}} \Rightarrow n = 17.3 \approx 18 \).

EXERCISE 7
We are given that \( X \) follows the normal distribution with \( \mu = 0 \) and \( \sigma^2 = 1 \). Therefore its probability density function (p.d.f.) is \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \), \(-\infty < x < \infty \). The random variable \( Y \) is said to follow the lognormal distribution if \( Y = e^X \) because \( \log(y) \) follows the normal distribution. To find its p.d.f we start with its cumulative distribution function which by definition is:

\[
F(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \log(y)) = F_X(\log(y)) \Rightarrow f(y) = \frac{1}{y} f_X(\log(y)) \Rightarrow \frac{1}{y} e^{-\frac{1}{2}(\log(y))^2}.
\]

EXERCISE 8
The radius of a circle, \( X \), is an exponential random variable with parameter \( \lambda \). Therefore its p.d.f. is \( f(x) = \lambda e^{-\lambda x} \). Let \( Y \) be the area of the circle. Then \( Y \sim \pi X^2 \). We want to find the p.d.f of \( Y \). We start with the c.d.f. function:

\[
F(y) = P(Y \leq y) = P(\pi X^2 \leq y) = P(X^2 \leq \frac{y}{\pi}) = P(-\sqrt{\frac{y}{\pi}} \leq X \leq \sqrt{\frac{y}{\pi}}) = F_X(\sqrt{\frac{y}{\pi}}) - F_X(-\sqrt{\frac{y}{\pi}}) \Rightarrow f(y) = F'(y) = \frac{1}{\sqrt{\frac{y}{\pi}}} f_X(\sqrt{\frac{y}{\pi}}) - 0 \Rightarrow f(y) = \frac{1}{\sqrt{\frac{y}{\pi}}} \lambda e^{-\lambda \sqrt{\frac{y}{\pi}}},
\]