University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Practice 2 - solutions

EXERCISE 1
This population has mean $\mu = 3$, and standard deviation $\sigma = 0.28$.

a. According to the central limit theorem $\bar{X}$ is distributed as $\bar{X} \sim N(3, \frac{0.28}{\sqrt{36}})$ or $\bar{X} \sim N(3, 0.047)$. We expect that the range $3 \pm 3(0.047)$ or 3 ± 0.141 or $(2.86, 3.141)$ will cover almost all the area. We conclude that the histogram should have been much narrower.

b. According to the central limit theorem $T \sim N(36(3), 0.28\sqrt{36})$ or $T \sim N(108, 1.68)$. The histogram should be centered around 108 with spread $(103, 113)$.

EXERCISE 2
It is given $X \sim N(2700, 400)$. The total supply for $n = 12$ weeks is $4000 + 12(2500) = 34000$. We want the supply to be below 2000 pounds or the total sugar use in these 12 weeks to be more than 32000 pounds:

$$P(T > 32000) = P\left(Z > \frac{32000 - 12(2700)}{400\sqrt{12}}\right) = P(Z > -0.29) = 0.6141.$$

EXERCISE 3
We know the the moment generating function of $N(\mu, \sigma)$ is $M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$.

a. Moment generating function of $X + Y$:

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{2\mu t + t^2\sigma^2}$$

Moment generating function of $X - Y$:

$$M_{X-Y}(s) = M_X(s)M_{-Y}(s) = e^{s^2\sigma^2}$$

b. Joint moment generating function of $X + Y, X - Y$:

$$\begin{aligned}
M_{X+Y,X-Y}(t,s) &= Ee^{(X+Y)t+(X-Y)s} \\
&= Ee^{X(t+s)+Y(t-s)} \\
&= M_X(t+s)M_Y(t-s) \\
&= e^{\mu(t+s)+\frac{1}{2}(t+s)^2\sigma^2} e^{\mu(t-s)+\frac{1}{2}(t-s)^2\sigma^2} \\
&= e^{2\mu t+t^2\sigma^2} e^{t^2\sigma^2} = M_{X+Y}(t)M_{X-Y}(s).
\end{aligned}$$

c. Since the joint moment generating function of $X + Y$ and $X - Y$ can be expressed as the product of the moment generating functions of $X + Y$ and $X - Y$ we conclude that $X + Y$ and $X - Y$ are independent.

EXERCISE 4

a. We can write $X_1 - 2X_2 + X_3$ as $\mathbf{a'X}$ where $\mathbf{a'} = (1, -2, 1)$. Therefore

$$var(\mathbf{aX}) = \mathbf{a'\Sigma a} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 18.$$

b. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \mathbf{AX}$. Therefore,

$$var(\mathbf{Y}) = \mathbf{A\Sigma A'} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 15 \\ 15 & 21 \end{pmatrix}.$$

EXERCISE 5

a. Let k be the minimum number of plays the casino must win. Then $1 \times k - (10000 - k) \times 35 > 400$. Solve for k to get $k = 9734$. Let Y be the number of games the casino must win, so $Y \sim b(10000, \frac{37}{38})$. The casino must win at least 9734 plays and this probability is $P(Y \geq 9734) = \sum_{y=9734}^{10000} \binom{10000}{y} \frac{37}{38}^y \frac{1}{38}^{10000-y}$. Using normal approximation to binomial: $P(Y \geq 9734) = P(Z > \frac{9733.5 - 10000\frac{37}{38}}{10000\frac{37}{38}\frac{1}{38}}) = P(Z > -0.21) = 0.5832$.

b. If we view the 10000 outcomes as a random sample from the following distribution then we can use the central limit theorem: Let $T = X_1 + \ldots X_{10000}$ be the sum of the 10000 outcomes.

X	$P(X)$
1	$\frac{37}{38}$
-35	$\frac{1}{38}$

This distribution has $\mu = 0.05263$ and $\sigma = 5.76$.

$P(T > 400) = P(Z > \frac{400-10000(0.05263)}{5.76\sqrt{10000}}) = P(Z > -0.22) = 0.5871$.

EXERCISE 6

We have $\mathbf{Z} = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\mu, \mathbf{\Sigma})$ and therefore the joint moment generating function of (X_i, Y_i) is $e^{\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t}}$. The joint moment generating function of $(\bar{X}, \bar{Y})$ is:

$$\begin{aligned} Ee^{t_1\bar{X}+t_2\bar{Y}} &= Ee^{t_1\frac{X_1+\ldots+X_n}{n}+t_2\frac{Y_1+\ldots+Y_n}{n}} \\ &= Ee^{t_1\frac{X_1}{n}+t_2\frac{Y_1}{n}} \times \ldots \times Ee^{t_1\frac{X_n}{n}+t_2\frac{Y_n}{n}}, \text{ because the pairs } (X_i, Y_i) \text{ are independent.} \end{aligned}$$

Each one of these expectations is the joint moment generating function of $\mathbf{AZ}$ with $\mathbf{A} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$ and $\mathbf{Z} = \begin{pmatrix} X \\ Y \end{pmatrix}$. Since $E(\mathbf{AZ}) = \frac{\boldsymbol{\mu}}{n}$ and $var(\mathbf{AZ}) = \frac{\mathbf{\Sigma}}{n^2}$, it follows that the joint moment generating function of $\mathbf{AZ}$ is $e^{\mathbf{t}'\frac{\boldsymbol{\mu}}{n} + \frac{1}{2}\mathbf{t}'\frac{\mathbf{\Sigma}}{n^2}\mathbf{t}}$. But we have n independent pairs, therefore te joint moment generating function of $(\bar{X}, \bar{Y})$ is $\left(e^{\mathbf{t}'\frac{\boldsymbol{\mu}}{n} + \frac{1}{2}\mathbf{t}'\frac{\mathbf{\Sigma}}{n^2}\mathbf{t}}\right)^n = e^{\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\frac{\mathbf{\Sigma}}{n}\mathbf{t}}$. This shows that the joint distribution of $(\bar{X}, \bar{Y})$ is bivariate normal $N_2(\boldsymbol{\mu}, \frac{\mathbf{\Sigma}}{n})$.

EXERCISE 7

We write $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \mathbf{AX}$, where $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3(\mathbf{0}, \mathbf{I})$ and $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$. Therefore, $var(\mathbf{Y}) = var(\mathbf{AX}) = \mathbf{AA'} = \mathbf{I_3}$, which means Y_1, Y_2, Y_3 are i.i.d. $N(0, 1)$.

EXERCISE 8

Let $\mathbf{X} = (X_1, X_2, X_3)$ has joint moment generating function

$$M_{\mathbf{X}}(t_1, t_2, t_3) = (1 - t_1 + 2t_2)^{-4}(1 - t_1 + 3t_3)^{-3}(1 - t_1)^{-2}.$$

Answer the following questions:

a. Find the moment generating function of (X_1, X_3).
$M_{X_1,X_3}(t_1, t_3) = M_{\mathbf{X}}(t_1, 0, t_3) = (1 - t_1)^{-6}(1 - t_1 + 3t_3)^{-3}$.

b. Find the moment generating function of X_1.
$M_{X_1}(t_1) = M_{\mathbf{X}}(t_1, 0, 0) = (1 - t_1)^{-9}$.

c. Find the moment generating function of X_3.
$M_{X_3}(t_3) = M_{\mathbf{X}}(0, 0, t_3) = (1 + 3t_3)^{-3}$.

d. Are X_1, X_3 independent?
No, because $M_{X_1,X_3}(t_1, t_3) \neq M_{X_1}(t_1) \times M_{X_3}(t_3)$.

e. Find the moment generating function of (X_2, X_3).
$M_{X_2,X_3}(t_2, t_3) = M_{\mathbf{X}}(0, t_2, t_3) = (1 + 2t_2)^{-4}(1 + 3t_3)^{-3}$.

f. Are X_2, X_3 independent?
$M_{X_2}(t_2) = M_{\mathbf{X}}(0, t_2, 0) = (1 + 2t_2)^{-4}$. Yes, X_2, X_3 are independent because $M_{X_2,X_3}(t_2, t_3) = M_{X_2}(t_2) \times M_{X_3}(t_3)$.