University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Practice 3

EXERCISE 1
Let $X_1, X_2, \cdots X_5$ be a random sample of size 5 from $N(0,1)$. Let X_6 be another independent observation from the same population.

a. What is the distribution of $W = \sum_{i=1}^{5} X_i^2$?

b. What is the distribution of $U = \sum_{i=1}^{5}(X_i - \bar{X})^2$?

c. What is the distribution of $\sum_{i=1}^{5}(X_i - \bar{X})^2 + X_6^2$?

EXERCISE 2
Let $Z_1, Z_2, \cdots, Z_{16}$ be a random sample of size 16 from the standard normal distribution $N(0,1)$.
Let $X_1, X_2, \cdots, X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$.
The two samples are independent.

a. Find $P(Z_1 > 2)$.

b. Find $P(\sum_{i=1}^{16} Z_i > 2)$.

c. Find $P(\sum_{i=1}^{16} Z_i^2 > 6.91)$.

d. Let S^2 be the sample variance of the first sample. Find c such that $P(S^2 > c) = 0.05$.

e. What is the distribution of Y, where $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64}(X_i - \mu)^2$?

f. Find EY.

g. Find $Var(Y)$.

h. Approximate $P(Y > 105)$.

EXERCISE 3
Let $X_1, X_2, \cdots, X_{25}$ be a random sample of size 25 from $N(6,2)$, and let $Y_1, Y_2, \cdots, Y_{35}$ be a random sample of size 35 from $\sim N(10,5)$. The two samples are independent.

a. Find the distribution of $X_1 + Y_1$.

b. Find $P(\bar{X} > 6.8)$.

c. What is the distribution of $\sum_{i=1}^{25}(\frac{X_i - 6}{2})^2$.

d. What is the distribution of W, where $W = \sum_{i=1}^{25}(\frac{X_i - 6}{2})^2 + \sum_{i=1}^{35}(\frac{Y_i - 10}{5})^2$?

e. Use the exact distribution of W to find b such that $P(W > b) = 0.05$.

f. Use the limiting distribution of W to approximate b such that $P(W > b) = 0.05$.

g. Let s_X^2 and s_Y^2 be the sample variances from the two samples above. Find c_1, c_2, c_3, c_4 such that the following expression follows the χ^2 distribution: $\frac{c_1 s_X^2}{c_3} + \frac{c_2 s_Y^2}{c_4}$

h. Let Q be a random variable with moment generating function $M_Q(t) = (1 - 2t)^{-20}$. If Q and W are independent, what is the distribution of $Q + W$?

i. Let U be another random variable with moment-generating function $M_U(t) = e^{500t + 5000t^2}$.
Find $P(27100 < (U - 500)^2 < 50200)$.

EXERCISE 4
Let X_1, X_2 be a random sample from a normal distribution with a mean of μ and a standard deviation σ. Show that $\frac{(n-1)s^2}{\sigma^2}$ has a χ^2 distribution with 1 degree of freedom. Hint: Find an expression of the sample variance in terms of X_1, X_2 and then substitute this expression into $\frac{(n-1)s^2}{\sigma^2}$.

EXERCISE 5
Let s_1^2 denote the sample variance for a random sample of 10 ln(LC50) values for copper and let s_2^2 denote the sample variance of 8 ln(LC50) values for lead, both samples using the same species of fish. The population variance for measurements on copper is assumed to be twice the corresponding population variance for measurements on lead. Find two numbers a and b such that $P(a < \frac{s_1^2}{s_2^2} < b) = 0.90$ assuming s_1^2 to be independent of s_2^2. Hint: You will find the following very useful: $F_{\alpha;n1,n2} = \frac{1}{F_{1-\alpha;n2,n1}}$.

EXERCISE 6
Suppose $X_1, X_2, \cdots X_5, X_6, X_7, W$ and U are as defined in exercise 1.

a. What is the distribution of $\frac{\sqrt{5}X_6}{\sqrt{W}}$?

b. What is the distribution of $\frac{2X_6}{\sqrt{U}}$?

c. What is the distribution of $\frac{2(X_6^2 + X_7^2)}{U}$?

EXERCISE 7

The *coefficient of variation* for a sample of values $X_1, X_2, \cdots, X_n$ is defined by $C.V. = \frac{s}{\bar{x}}$, where s is the sample standard deviation and $\bar{x}$ is the sample mean. This term gives the standard deviation as a proportion of the mean, and it is sometimes an informative quantity. For example a value of $s = 10$ has little meaning unless we can compare it with something else. If $s = 10$ and $\bar{x} = 1000$ the amount of variation is small relative to the mean. However, if $s = 10$ and $\bar{x} = 5$ then the variation is quite large relative to the mean. If we were studying the precision (variation in repeated measurements) of a measuring instrument, the first case $C.V. = \frac{10}{1000}$ might give quite acceptable precision but the second case $C.V. = \frac{10}{5}$ would be quite unacceptable. Let $X_1, X_2, \cdots, X_{10}$ denote a random sample of size 10 from a normal distribution with mean 0 and variance σ^2.

a. Let $W = \frac{10\bar{x}^2}{s^2}$. Find the distribution of W.

b. Let $Y = \frac{s^2}{10\bar{x}^2}$. Find the distribution of Y.

c. Find the number c such that $P(-c < \frac{s}{\bar{x}} < c) = 0.95$.

d. Find c such that $P(Y > c) = 0.2$.

e. Determine the value of c such that $P(W > c) = 0.95$.

EXERCISE 8

Let $Z_1, Z_2, \cdots, Z_{16}$ be a random sample of size 16 from the standard normal distribution $N(0, 1)$.
Let $X_1, X_2, \cdots, X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$.
The two samples are independent.

a. If $Y \sim \chi^2_{80}$, find c such that

$$c\frac{\sum_{i=1}^{16} Z_i^2}{Y} \sim F_{16,80}.$$

b. Let $Q \sim \chi^2_{60}$. Find c such that

$$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95.$$

c. Use the t table to find the 80_{th} percentile of the $F_{1,30}$ distribution.

d. Find c such that $P(F_{60,20} > c) = 0.99$.