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Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Practice 3 - solutions

EXERCISE 1
We are given that $X_1, X_2, \cdots X_5$ and X_6 is a random sample from the standard normal distribution $N(0,1)$. We will also need the following result: If X, Y are independent with $X \sim \chi^2_n$ and $Y \sim \chi^2_m$ then $X+Y \sim \chi^2_{n+m}$.

a. $W = \sum_{i=1}^5 X_i^2 \sim \chi^2_5$.

b. $U = \sum_{i=1}^5 (X_i - \bar{X})^2 = \frac{(5-1)S^2}{1} \sim \chi^2_4$.

c. $\sum_{i=1}^5 (X_i - \bar{X})^2 + X_6^2 = \frac{(5-1)S^2}{1} + X_6^2 \sim \chi^2_5$.

EXERCISE 2
Let $Z_1, Z_2, \cdots, Z_{16}$ be a random sample of size 16 from the standard normal distribution $N(0,1)$.
Let $X_1, X_2, \cdots, X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$.
The two samples are independent.

a. Find $P(Z_1 > 2)$. *Answer:* $P(Z_1 > 2) = 1 - 0.9772 = 0.0228$ (from the standard normal table).

b. Find $P(\sum_{i=1}^{16} Z_i > 2)$.
Answer: $P(\sum_{i=1}^{16} Z_i > 2) = P(Z > \frac{2-16(0)}{1\sqrt{16}}) = P(Z > 0.5) = 1 - 0.6915 = 0.3085$.

c. Find $P(\sum_{i=1}^{16} Z_i^2 > 6.91)$. *Answer:* $P(\sum_{i=1}^{16} Z_i^2 > 6.91) = P(\chi^2_{16} > 6.91) = 1 - 0.025 = 0.975$.

d. Let S^2 be the sample variance of the first sample. Find c such that $P(S^2 > c) = 0.05$.
Answer: $P(S^2 > c) = 0.05$ or $P(\frac{(16-1)S^2}{1^2} > \frac{15c}{1^2}) = 0.05$ or $P(\chi^2_{15} > 15c) = 0.05$. From the χ^2 table: $15c = 25.00$ or $c = \frac{5}{3}$.

e. What is the distribution of Y, where $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2$?
Answer: $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2 = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (\frac{X_i - \mu}{1})^2 = \chi^2_{16} + \chi^2_{64} = \chi^2_{80}$.

f. Find EY. *Answer:* $EY = E(\chi^2_{80}) = 80$.

g. Find $Var(Y)$. *Answer:* $Var(Y) = Var(\chi^2_{80}) = 2(80) = 160$.

h. Approximate $P(Y > 105)$.
Answer: $P(Y > 105) = P(Z > \frac{105-80}{\sqrt{160}}) = P(Z > 1.98) = 1 - 0.9761 = 0.0239$.

EXERCISE 3

Let $X_1, X_2, \cdots, X_{25}$ be a random sample of size 25 from $N(6,2)$, and let $Y_1, Y_2, \cdots, Y_{35}$ be a random sample of size 35 from $\sim N(10,5)$. The two samples are independent.

a. Find the distribution of $X_1 + Y_1$.
   Answer: $X_1 + Y_1 \sim N(16, \sqrt{29})$.

b. Find $P(\bar{X} > 6.8)$.
   Answer: $P(\bar{X} > 6.8) = P(Z > \frac{6.8-6}{\frac{2}{\sqrt{25}}} = P(Z > 2) = 1 - 0.9772 = 0.0228$.

c. What is the distribution of $\sum_{i=1}^{25}(\frac{X_i-6}{2})^2$.
   Answer: χ^2_{25} (chi square with 25 degrees of freedom).

d. What is the distribution of W, where $W = \sum_{i=1}^{25}(\frac{X_i-6}{2})^2 + \sum_{i=1}^{35}(\frac{Y_i-10}{5})^2$?
   Answer: $\chi^2_{25} + \chi^2_{35} = \chi^2_{60}$ (chi square with 60 degrees of freedom).

e. Use the exact distribution of W to find b such that $P(W > b) = 0.05$.
   Answer: From the chi square table using $P(W < b) = 0.95$ we find $b = 79.08$.

f. Use the limiting distribution of W to approximate b such that $P(W > b) = 0.05$.
   Answer: Using $Z = \frac{W-60}{\sqrt{120}}$ we get $1.645 = \frac{b-60}{\sqrt{120}} \Rightarrow b = 78.02$.

g. Let s^2_X and s^2_Y be the sample variances from the two samples above. Find c_1, c_2, c_3, c_4 such that the following expression follows the χ^2 distribution: $\frac{c_1 s^2_X}{c_3} + \frac{c_2 s^2_X}{c_4}$
   Answer: $c_1 = 24, c_2 = 34, c_3 = 4, c_4 = 25$ and the distribution will χ^2_{58}.

h. Let Q be a random variable with moment generating function $M_Q(t) = (1-2t)^{-20}$. If Q and W are independent, what is the distribution of $Q + W$?
   Answer: Q follows χ^2_{40}, and therefore $Q + W \sim \chi^2_{100}$.

i. Let U be another random variable with moment-generating function $M_U(t) = e^{500t+5000t^2}$.
   Find $P(27100 < (U-500)^2 < 50200)$.
   Answer: From the moment generating function we know that $U \sim N(500, 100)$. Therefore,

$$P\left[\frac{27100}{10000} < \left(\frac{U-500}{100}\right)^2 < \frac{50200}{10000}\right] = P(2.71 < \chi^2_1 < 5.02) = 0.975 - 0.90 = 0.075.$$

EXERCISE 4

In the case when $n = 2$ the sample mean is $\bar{X} = \frac{X_1+X_2}{2}$ and therefore the sample varianceS^2 is:
$S^2 = \frac{1}{2-1}\sum_{i=1}^{2}(X_i - \bar{X})^2 = [X_1 - \frac{1}{2}(X_1+X_2)]^2 + [X_2 - \frac{1}{2}(X_1+X_2)]^2 = [\frac{1}{2}(X_1-X_2)]^2 + [\frac{1}{2}(X_2-X_1)]^2 = 2[\frac{1}{2}(X_1-X_2)]^2 = \frac{(X_1-X_2)^2}{2}$. It follows that when $n = 2$, $\frac{(n-1)S^2}{\sigma^2} = \frac{(X_1-X_2)^2}{2\sigma^2} = (\frac{X_1-X_2}{\sqrt{2}\sigma})^2$. We must show now that this quantity is equal to the square of a standard normal variable (Z^2) which we know that follows the chi-square distribution with 1 degree of freedom. Because $X_1 - X_2$ is a linear combination of independent normally distributed random variables we can find that the mean of $X_1 - X_2$ is $E(X_1 - X_2) = \mu - \mu = 0$ and its variance is $Var(X_1 - X_2) = Var(X_1) + Var(X_2) = 2\sigma^2$ (the covariance is zero because X_1, X_2 are independent). Therefore $Z = \frac{X_1-X_2-0}{\sqrt{2}\sigma}$ has a standard normal distribution. Its square then is χ^2_1. But $Z^2 = \frac{(n-1)S^2}{\sigma^2}$ which means that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_1$. This completes the proof.

EXERCISE 5

We have 2 samples: $n_1 = 10$ with sample variance S_1^2, and $n_2 = 8$ with sample variance S_2^2. We want to find a and b such that $P(a < \frac{s_1^2}{s_2^2} < b) = 0.90$. We must transform the ratio of the 2 sample variances into F. This is done by dividing each sample variance with the corresponding population variance. We know that the population variance for measurements on copper is twice the corresponding population variance for masurements on lead. Therefore $\sigma_1^2 = 2\sigma_2^2$. Then the above probability statement can be written as $P(\frac{a}{2} < \frac{s_1^2}{2s_2^2} < \frac{b}{2}) = 0.90$ or $P(\frac{a}{2} < F_{9,7} < \frac{b}{2}) = 0.90$ or If we choose $\frac{a}{2}$ and $\frac{b}{2}$ to be 2 points such that the area under the $F_{9,7}$ below $\frac{a}{2}$ is 0.05 and the area below $\frac{b}{2}$ is 0.95 and using $F_{0.05;9,7} = \frac{1}{F_{1-0.05;7,9}}$ we get: $F_{0.95;9,7} = 3.68 \Rightarrow \frac{b}{2} = 3.68 \Rightarrow b = 7.36$. To find a we need first to find $F_{0.05;9,7} = \frac{1}{F_{1-0.05;7,9}} = \frac{1}{3.29} = 0.304$. Therefore $\frac{a}{2} = 0.304 \Rightarrow a = 0.608$.

EXERCISE 6

We need to use the definition of the t and F distribution.

a. $\frac{\sqrt{5}X_6}{\sqrt{W}} = \frac{X_6}{\sqrt{W/5}} \sim t_5$. This is the ratio of a standard normal ($X_6 \sim N(0,1)$) with the square root of a chi-square random variable divided by its degrees of freedom ($W \sim \chi_5^2$).

b. $\frac{2X_6}{\sqrt{U}} = \frac{X_6}{\sqrt{U/4}} \sim t_4$.

c. $\frac{2(X_6^2+X_7^2)}{U} = \frac{\frac{2(X_6^2+X_7^2)}{4}}{\frac{U}{4}} = \frac{\frac{X_6^2+X_7^2}{2}}{\frac{U}{4}} \sim F_{2,4}$.

EXERCISE 7

a. $$W = \frac{10\bar{x}^2}{s^2} = \frac{\frac{\left(\frac{\bar{X}-0}{\frac{\sigma}{\sqrt{10}}}\right)^2}{1}}{\frac{\frac{9S^2}{\sigma^2}}{9}} \Rightarrow W \sim F_{1,9}.$$

b. $$Y \sim F_{9,1}.$$

c. $$P\left(-c < \frac{s}{\bar{x}} < c\right) = 0.95 \Rightarrow P\left(\frac{S^2}{\bar{X}^2} < c^2\right) = 0.95$$

Or

$$P\left(\frac{S^2}{10\bar{X}^2} < \frac{c^2}{10}\right) = 0.95 \Rightarrow P\left(F_{9,1} < \frac{c^2}{10}\right) = 0.95 \Rightarrow \frac{c^2}{10} = 240.5 \text{ (from the } F \text{ table).}$$

Therefore, $c = 49.04$.

d. $$P(Y > c) = 0.20 \Rightarrow P(F_{9,1} > c) = 0.20 \Rightarrow P(\frac{1}{F_{9,1}} < \frac{1}{c}) = 0.20 \Rightarrow P(F_{1,9} < \frac{1}{c}) = 0.20.$$

Therefore, from t table $\Rightarrow \frac{1}{c} = (t_{0.60;9})^2 = (0.261)^2 \Rightarrow c = 14.68$.

e. $$P(W > c) = 0.95 \text{ (} c \text{ is the 5th percentile of } F_{1,9}\text{).}$$

Therefore,

$$c = \frac{1}{F_{0.95;9,1}} = \frac{1}{240.5} = 0.00416.$$

EXERCISE 8
Let $Z_1, Z_2, \cdots, Z_{16}$ be a random sample of size 16 from the standard normal distribution $N(0,1)$.
Let $X_1, X_2, \cdots, X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$.
The two samples are independent.

a. Find c such that

$$c\frac{\sum_{i=1}^{16} Z_i^2}{Y} \sim F_{16,80}.$$

Answer:

$$\frac{\frac{\sum_{i=1}^{16} Z_i^2}{16}}{\frac{Y}{80}} \sim F_{16,80}$$

Or $\frac{80}{16}\frac{\chi^2_{16}}{\chi^2_{80}} \sim F_{16,80}$ or $5\frac{\chi^2_{16}}{\chi^2_{80}} \sim F_{16,80}$. Therefore $c = 5$.

b. Let $Q \sim \chi^2_{60}$. Find c such that

$$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95.$$

Answer:

$$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95 \text{ or } P\left(\frac{Z_1}{\sqrt{\frac{Q}{60}}} < \sqrt{60}c\right) = 0.95.$$

Or $P(t_{60} < \sqrt{60}c) = 0.95$. From the t table $\sqrt{60}c = 1.671$ or $c = 0.22$.

c. Use the t table to find the 80_{th} percentile of the $F_{1,30}$ distribution.
Answer: $F_{0.80;1,30} = (t_{0.90;30})^2 = 1.310^2 = 1.72$.

d. Find c such that $P(F_{60,20} > c) = 0.99$.
Answer: We want to find c which is the $1st$ percentile of the $F_{60,20}$ distribution. This is equal to: $F_{0.01;60,20} = \frac{1}{F_{0.99;20,60}} = \frac{1}{2.20} = 0.45$.