**University of California, Los Angeles**
**Department of Statistics**

**Statistics 100B** **Instructor: Nicolas Christou**

# Practice 4

**EXERCISE 1**
Suppose $X_1, X_2, \cdots X_n$ is a random sample from $Bernoulli(p)$.

a. Show that the mle of $p$ is $\hat{p} = \frac{X}{n}$.

b. Show that the mle of part (a) attains the Cramer-Rao lower bound.

c. If $n = 10$ and $X = 5$, plot the log likelihood function.

**EXERCISE 2**
Suppose that $X$ follows a geometric distribution

$$P(X = x) = p(1-p)^{x-1}$$

and assume an i.i.d. sample of size $n$. Find the mle of $p$.

**EXERCISE 3**
Given a random sample of size $n$ from a population which has known mean $\mu$ and unknown variance $\sigma^2$, show that $\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$ is an unbiased estimator of the true but unknown variance $\sigma^2$.

**EXERCISE 4**
If $X_1, X_2$ and $X_3$ are a random sample from a normal population with mean $\mu$ and variance $\sigma^2$, what is the relative efficiency of the estimator $\hat{\mu} = \frac{X_1+2X_2+X_3}{4}$ with respect to the estimator $\bar{X} = \frac{X_1+X_2+X_3}{3}$.

**EXERCISE 5**
If $\bar{X}_1$ is the mean of a random sample of size $n$ from $N \sim N(\mu, \sigma_1)$ and $\bar{X}_2$ is the mean of a random sample of size $n$ from $N \sim N(\mu, \sigma_2)$, and the two samples are independent, show that

a. $w\bar{X}_1 + (1-w)\bar{X}_2$, where $0 \leq w \leq 1$, is an unbiased estimator of $\mu$.

b. the variance of this estimator is minimum when $w = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

**EXERCISE 6**
If $X_1, X_2, \cdots, X_n$ are the values of a random sample of size $n$ from a Poisson population with parameter $\lambda$, find an estimate of $\lambda$ using the the method of maximum likelihood.

**EXERCISE 7**
If $X$ has a binomial distributon with $n$ trials and success probability $p$, show that $\frac{X}{n}$ is a consistent estimator of $p$.

**EXERCISE 8**
Suppose that $X_1, X_2, \cdots, X_n$ represents a random sample of size $n$ from a normal population with mean $\mu$ and variance $\sigma^2$. Show that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is a consistent estimator of $\sigma^2$. Hint: Can the $\chi^2$ distribution help you in finding the variance of $S^2$?

**EXERCISE 9**

If $X$ is binomial $(n, p)$, then the variance of $\hat{p} = \frac{X}{n}$ (which is the maximum likelihood estimate of $p$) is $\sigma^2_{\hat{p}} = \frac{p(1-p)}{n}$. This variance is often estimated by $\hat{\sigma}^2 = \frac{\hat{p}(1-\hat{p})}{n} = \frac{\frac{X}{n}(1-\frac{X}{n})}{n}$. Is this an unbiased estimator of $\sigma^2_{\hat{p}}$? If not find a constant $c$ so that $c\hat{\sigma}^2$ is unbiased.

**EXERCISE 10**

The numbers $w_1, w_2, \cdots, w_n$ are known positive values. The random variables $X_1, X_2, \cdots, X_n$ are independent, and the distribution of $X_i$ is $N(\mu, \frac{\sigma}{\sqrt{w_i}})$. Both parameters $\mu$ and $\sigma$ are unknown. Find the maximum likelihood estimates of $\mu$ and $\sigma^2$.

**EXERCISE 11**

Let $X_1, X_2, \cdots, X_n$ be an i.i.d. random sample from $N(\mu, \sigma)$.

a. Which of the following estimates is unbiased? Show all your work.

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}, \quad S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

b. Which of the estimates of part (a) has the smaller $MSE$?

**EXERCISE 12**

Let $X_1, X_2, \cdots, X_n$ be an i.i.d. random sample from a normal population with mean zero and unknown variance $\sigma^2$.

a. Find the maximum likelihood estimate of $\sigma^2$.

b. Show that the estimate of part (a) is unbiased estimator of $\sigma^2$.

c. Find the variance of the estimate of part (a). Is it consistent?

d. Show that the variance of the estimate of part (a) is equal to the Cramer-Rao lower bound.